PERVERSE COHERENT SHEAVES AND FOURIER-MUKAI TRANSFORMS ON SURFACES

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Abstract. We study perverse coherent sheaves on the resolution of rational double points. As examples, we consider rational double points on 2-dimensional moduli spaces of stable sheaves on $K3$ and elliptic surfaces. Then we show that perverse coherent sheaves appears in the theory of Fourier-Mukai transforms. As an application, we generalize the Fourier-Mukai duality for $K3$ surfaces to our situation.

Contents

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0. Introduction

Let $\pi : X \to Y$ be a birational map such that $\dim \pi^{-1}(y) \leq 1, y \in Y$. Then Bridgeland [Br3] introduced the abelian category ${}^p\operatorname{Per}(X/Y)(\subset \mathbf{D}(X))$ of perverse coherent sheaves in order to show that flops of smooth 3-folds preserves the derived categories of coherent sheaves. By using the moduli of perverse coherent sheaves on X, Bridgeland constructed the flop $X' \to Y$ of $X \to Y$. Then the Fourier-Mukai transform by the universal family induces an equivalence $\mathbf{D}(X) \cong \mathbf{D}(X')$. In [VB], Van den Bergh showed that ${}^p\operatorname{Per}(X/Y)$ is Morita equivalent to the category $\operatorname{Coh}_{\mathcal{A}}(Y)$ of $\mathcal{A}$-modules on Y and gave a different proof of Bridgeland result, where $\mathcal{A}$ is a sheaf of (non-commutative) algebras over Y. Although the main examples of the birational contraction are small contraction of 3-folds, 2-dimensional cases seem to be still interesting. In [NY1], [NY2], Nakajima and the author studied perverse coherent sheaves for the blowing up $X \to Y$ of a smooth surface Y at a point. In this case, by analysing wall-crossing phenomena, we related the moduli of stable perverse coherent sheaves to the moduli of usual stable sheaves. Next example is the minimal resolution of a rational double point. Let G be a finite subgroup of $SU(2)$ acting on $\mathbb{C}^2$ and set $Y := \mathbb{C}^2/G$. Let $\pi : X \to Y$ be the resolution of Y. Then the relation between the perverse coherent sheaves and the usual coherent sheaves on X are discussed by Nakajima. Their moduli spaces are constructed as Nakajima's quiver varieties [N1] and their differences are described by the wall crossing phenomena [N2]. Toda [T] also treated special cases. In this paper, we are interested in the global case. Thus we consider the minimal resolution $\pi : X \to Y$ of a normal projective surface Y with rational double points as singularities.

As examples, we shall show that perverse coherent sheaves naturally appear if we consider the Fourier-Mukai transforms on $K3$ and elliptic surfaces. In our previous paper [Y5], we studied Fourier-Mukai transforms defined by the moduli spaces of (semi)-stable sheaves Y' on X. Our assumption is the genericity of the polarization. If the polarization is not general, then Y' is singular at properly semi-stable sheaves. In this case, we still have the Fourier-Mukai transform by using the resolution X' of Y'. Then the category of perverse coherent sheaves on X' naturally appears. In particular, we show that the universal family on $X' \times X$ is the universal family of stable perverse coherent sheaves on X' (Theorem 3.6.1). Thus we have a kind of duality between X and X', which is a generalization of the relation between an abelian variety and its dual. We call this kind of duality *Fourier-Mukai duality*. The Fourier-Mukai duality for a $K3$ surface was studied by Bartocci, Bruzzo, Hernández Ruipérez [BBH], Mukai [Mu3], Orlov [O], Bridgeland [Br2], and was first proved by Huybrechts in [H] under the genericity of the polarization. He also proved that the Fourier-Mukai transform preserves nice abelian subcategories. We also give a generalization of this result (Theorem 3.5.8). Then we can generalize the result on the preservation of stability by the Fourier-Mukai transform in [Y5] to our situation.

For the relative Fourier-Mukai transforms on elliptic surfaces, we also get similar results. Let G be a finite group acting on a projective surface X. Assume that K_X is the pull-back of a line bundle on $Y := X/G$. Then the McKay correspondence [VB] implies that $\operatorname{Coh}_G(X)$ is equivalent to ${}^{-1}\operatorname{Per}(X'/Y)$, where $X' \to Y$ is the minimal resolution of Y. The equivalence is given by a Fourier-Mukai transform associated to a moduli space of stable G-sheaves of dimension 0. If X is a $K3$ surface or an abelian surface, then we have many 2-dimensional moduli spaces of stable G-sheaves. We also treat the Fourier-Mukai transform induced by the moduli of G-sheaves.

In section 1, we consider an abelian subcategory $\mathcal{C}$ of $\mathbf{D}(X)$ which is Morita equivalent to $\operatorname{Coh}_{\mathcal{A}}(Y)$, where $\pi : X \to Y$ be a birational contraction from a smooth variety X and $\mathcal{A}$ is a sheaf of (non-commutative) algebras over Y. We call an object of $\mathcal{C}$ a perverse coherent sheaf. Since ${}^{-1}\operatorname{Per}(X/Y)$ is Morita equivalent to $\operatorname{Coh}_{\mathcal{A}}(Y)$ for an algebra $\mathcal{A}$ [VB], our definition is compatible with Bridgeland's definition. We also study irreducible objects and local projective generators of $\mathcal{C}$. As examples, we shall give generalizations of ${}^p\operatorname{Per}(X/Y)$, $p = -1, 0$. We next explain families of perverse coherent sheaves and the relative version of Morita equivalence. Then we can use Simpson's moduli spaces of stable $\mathcal{A}$-modules [S] to construct the moduli spaces of stable perverse coherent sheaves. Since Simpson's stability is not good enough for the 0-dimensionional objects, we also introduce a refinement of the stability and construct the moduli space, which is close to King's stability [K].

In section 2, we study perverse coherent sheaves on the resolution of rational double points. We first introduce two kind of categories $\operatorname{Per}(X/Y, \mathbf{b}_1, \dots, \mathbf{b}_n)$ and $\operatorname{Per}(X/Y, \mathbf{b}_1, \dots, \mathbf{b}_n)^*$ associated to a sequence of line bundles on the exceptional curves of the resolution of rational singularities and show that they are the category of perverse coherent sheaves in the sense in section 1. They are generalizations of ${}^{-1}\operatorname{Per}(X/Y)$ and ${}^0\operatorname{Per}(X/Y)$ respectively.

We next study the moduli of 0-dimensional objects on the resolution of rational double points. We introduce the wall and the chamber structure and study the Fourier-Mukai transforms induced by the moduli spaces. Under a suitable stability condition for $\mathbb{C}_x$, $x \in X$, we show that the category of perverse coherent sheaves is equivalent to ${}^{-1}\operatorname{Per}(X/Y)$ (cf. Proposition 2.4.7). We also construct local projective generators under suitable conditions.

In section 3, we consider the Fourier-Mukai transforms on $K3$ surfaces. We generalize known facts on the 2-dimensional moduli spaces of usual stable sheaves to those of stable objects. Then we define similar categories $\mathfrak{A}$ and $\mathfrak{A}^\mu$ to those in [Br4], and generalize results in [H]. In particular, we study the relation of Fourier-Mukai transforms and the categories $\mathfrak{A}, \mathfrak{A}^\mu$ (Theorem 3.5.8). This result will be used to study Bridgeland's stable objects in [MYY]. We also prove the Fourier-Mukai duality (Theorem 3.6.1). Finally we give some conditions for the preservation of Gieseker stability conditions. Fourier-Mukai transforms on elliptic surfaces and Fourier-Mukai transforms by equivariant coherent sheaves are treated in sections 4 and 5.

Notation.

(i) For a scheme X, $\mathrm{Coh}(X)$ denotes the category of coherent sheaves on X and $\mathbf{D}(X)$ the bounded derived category of $\mathrm{Coh}(X)$. We denote the Grothendieck group of X by $K(X)$.

(ii) Let $\mathcal{A}$ be a sheaf of $\mathcal{O}_X$-algebras on a scheme X which is coherent as an $\mathcal{O}_X$-module. Let $\mathrm{Coh}_{\mathcal{A}}(X)$ be the category of coherent $\mathcal{A}$-modules on X and $\mathbf{D}_{\mathcal{A}}(X)$ the bounded derived category of $\mathrm{Coh}_{\mathcal{A}}(X)$.

(iii) Assume that X is a smooth projective variety. Let E be an object of $\mathbf{D}(X)$. $E^\vee := \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(E, \mathcal{O}_X)$ denotes the dual of E. We denote the rank of E by $\operatorname{rk} E$. For a fixed nef divisor H on X, $\deg(E)$ denotes the degree of E with respect to H. For $G \in K(X)$, $\operatorname{rk} G > 0$, we also define the twisted rank and degree by $\operatorname{rk}_G(E) := \operatorname{rk}(G^\vee \otimes E)$ and $\deg_G(E) := \deg(G^\vee \otimes E)$ respectively. We set $\mu_G(E) := \deg_G(E)/\operatorname{rk}_G(E)$, if $\operatorname{rk} E \neq 0$.

(iv) **Integral functor.** For two schemes X, Y and an object $\mathcal{E} \in \mathbf{D}(X \times Y)$, $\Phi^{\mathcal{E}}_{X \to Y} : \mathbf{D}(X) \to \mathbf{D}(Y)$ is the integral functor

$$\Phi^{\mathcal{E}}_{X \to Y}(E) := \mathbf{R}p_{Y*}(\mathcal{E} \overset{\mathbf{L}}{\otimes} p_X^*(E)),\ E \in \mathbf{D}(X), \tag{0.1}$$

where $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are projections. If $\Phi^{\mathcal{E}}_{X \to Y}$ is an equivalence, it is said to be the *Fourier-Mukai transform*.

(v) $\mathbf{D}(X)_{op}$ denotes the opposit category of $\mathbf{D}(X)$. We have a functor

$$\begin{array}{cccc} D_X: & \mathbf{D}(X) & \to & \mathbf{D}(X)_{op} \\ & E & \mapsto & E^\vee. \end{array}$$

(vi) Assume X is a smooth projective surface.

(a) We set $H^{ev}(X, \mathbb{Z}) := \bigoplus_{i=0}^2 H^{2i}(X, \mathbb{Z})$. In order to describe the element x of $H^{ev}(X, \mathbb{Z})$, we use two kinds of expressions: $x = (x_0, x_1, x_2) = x_0 + x_1 + x_2 \varrho_X$, where $x_0 \in \mathbb{Z}, x_1 \in H^2(X, \mathbb{Z}), x_2 \in \mathbb{Z}$, and $\int_X \varrho_X = 1$. For $x = (x_0, x_1, x_2)$, we set $\operatorname{rk} x := x_0$ and $c_1(x) = x_1$.

(b) We define a homomorphism

$$\begin{array}{cccc} \gamma: & K(X) & \to & \mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z} \\ & E & \mapsto & (\operatorname{rk} E, c_1(E), \chi(E)) \end{array} \tag{0.2}$$

and set $K(X)_{\mathrm{top}} := K(X)/\ker \gamma$. We denote $E \mod \ker \gamma$ by $\tau(E)$. $K(X)_{\mathrm{top}}$ has a bilinear form $\chi(\ ,\ )$.

(c) **Mukai lattice.** We define a lattice structure $\langle\ ,\ \rangle$ on $H^{ev}(X, \mathbb{Z})$ by

$$\begin{aligned} \langle x, y \rangle :=& -\int_X x^\vee \cup y \\ =&(x_1, y_1) - (x_0 y_2 + x_2 y_0), \end{aligned} \tag{0.3}$$

where $x = (x_0, x_1, x_2)$ (resp. $y = (y_0, y_1, y_2)$) and $x^\vee = (x_0, -x_1, x_2)$. It is now called the *Mukai lattice*. Mukai lattice has a weight-2 Hodge structure such that the (p, q)-part is $\bigoplus_i H^{p+i, q+i}(X)$. We set

$$\begin{aligned} H^{ev}(X, \mathbb{Z})_{\mathrm{alg}} =& H^{1,1}(H^{ev}(X, \mathbb{C})) \cap H^{ev}(X, \mathbb{Z}) \\ \cong& \mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z}. \end{aligned} \tag{0.4}$$

Let E be an object of $\mathbf{D}(X)$. If X is a $K3$ surface or $\operatorname{rk} E = 0$, we define the *Mukai vector* of E as

$$v(E) := \operatorname{rk}(E) + c_1(E) + (\chi(E) - \operatorname{rk}(E))\varrho_X \in H^{ev}(X, \mathbb{Z}). \tag{0.5}$$

Then for $E, F \in \mathbf{D}(X)$ such that the Mukai vectors are well-defined, we have

$$\chi(E, F) = -\langle v(E), v(F) \rangle. \tag{0.6}$$

(d) Since $\deg_G(E)$ is determined by the Chern character $\mathrm{ch}(E)$, we can also define $\deg_G(v)$, $v \in H^{ev}(X, \mathbb{Z})_{\mathrm{alg}}$ by using $E \in \mathbf{D}(X)$ with $v(E) = v$.

1.1. **Tilting and Morita equivalence.** Let X be a smooth projective variety and $\pi : X \to Y$ a birational map. Let $\mathcal{O}_Y(1)$ be an ample line bundle on Y and $\mathcal{O}_X(1) := \pi^*(\mathcal{O}_Y(1))$. We are interested in a subcategory $\mathcal{C}$ of $\mathbf{D}(X)$ such that

(i) $\mathcal{C}$ is the heart of a bounded t-structure of $\mathbf{D}(X)$.
(ii) There is a local projective generator G of $\mathcal{C}$ [VB]:
  (a) $\mathbf{R}\pi_*\mathbf{R}\mathcal{H}om(G,E) \in \operatorname{Coh}(Y)$ for all $E \in \mathcal{C}$ and
  (b) $\mathbf{R}\pi_*\mathbf{R}\mathcal{H}om(G,E) = 0$, $E \in \mathcal{C}$ if and only if $E = 0$.

By these properties, we get

$$\mathcal{C} = \{E \in \mathbf{D}(X) | \mathbf{R}\pi_*\mathbf{R}\mathcal{H}om(G,E) \in \operatorname{Coh}(Y)\}. \tag{1.1}$$

Definition 1.1.1. (1) A *perverse coherent sheaf* E is an object of $\mathcal{C}$. $\mathcal{C}$ is the *category of perverse coherent sheaves.*
(2) For $E \in \mathbf{D}(X)$, ${}^pH^i(E) \in \mathcal{C}$ denotes the i-th cohomology object of E with respect to the t-structure.

The following is an easy consequence of the properties (a), (b) of G. For a convenience sake, we give a proof.

Lemma 1.1.2. *Let G be a local projective generator of $\mathcal{C}$.*
(1) For $E \in \mathcal{C}$, there is a locally free sheaf V on Y and a surjective morphism

$$\phi : \pi^*(V) \otimes G \to E \tag{1.2}$$

in $\mathcal{C}$. In particular, we have a resolution

$$\cdots \to \pi^*(V_{-1}) \otimes G \to \pi^*(V_0) \otimes G \to E \to 0 \tag{1.3}$$

of E such that V_i, $i \le 0$ are locally free sheaves on Y.
(2) Let $G' \in \mathcal{C}$ be a local projective object of $\mathcal{C}$: $\mathbf{R}\pi_*\mathbf{R}\mathcal{H}om(G',E) \in \operatorname{Coh}(Y)$ for all $E \in \mathcal{C}$. If G is a locally free sheaf, then G' is also a locally free sheaf

Proof. (1) By the property (a) of G, we can take a morphism $\varphi : V \to \mathbf{R}\pi_*\mathbf{R}\mathcal{H}om(G,E)$ in $\mathbf{D}(Y)$ such that $V \to H^0(\mathbf{R}\pi_*\mathbf{R}\mathcal{H}om(G,E))$ is surjective in $\operatorname{Coh}(Y)$. Since

$$\begin{aligned}\operatorname{Hom}(\mathbf{L}\pi^*(\mathbf{R}\pi_*\mathbf{R}\mathcal{H}om(G,E)) \otimes G, E) &= \operatorname{Hom}(\mathbf{L}\pi^*(\mathbf{R}\pi_*\mathbf{R}\mathcal{H}om(G,E)), \mathbf{R}\mathcal{H}om(G,E)) \\ &= \operatorname{Hom}(\mathbf{R}\pi_*\mathbf{R}\mathcal{H}om(G,E), \mathbf{R}\pi_*\mathbf{R}\mathcal{H}om(G,E)),\end{aligned} \tag{1.4}$$

we have a morphism $\phi : \pi^*(V) \otimes G \to E$ such that the induced morphism $V \to \mathbf{R}\pi_*\mathbf{R}\mathcal{H}om(G, \pi^*(V) \otimes G) \to \mathbf{R}\pi_*\mathbf{R}\mathcal{H}om(G,E)$ is φ. Then $\operatorname{coker}\phi \in \mathcal{C}$ satisfies $\mathbf{R}\pi_*\mathbf{R}\mathcal{H}om(G, \operatorname{coker}\phi) = 0$. By our assumption on G, $\operatorname{coker}\phi = 0$. Thus ϕ is surjective in $\mathcal{C}$.

(2) We take a surjective homomorphism (1.2) for G'. Let U be an affine open subset of Y. We note that

$$\operatorname{Hom}(G'_{|\pi^{-1}(U)}, \ker\phi_{|\pi^{-1}(U)}[1]) = H^1(U, \mathbf{R}\pi_*\mathbf{R}\mathcal{H}om(G'_{|\pi^{-1}(U)}, \ker\phi_{|\pi^{-1}(U)})) = 0. \tag{1.5}$$

Hence

$$\operatorname{Hom}(G'_{|\pi^{-1}(U)}, \pi^*(V) \otimes G_{|\pi^{-1}(U)}) \to \operatorname{Hom}(G'_{|\pi^{-1}(U)}, G'_{|\pi^{-1}(U)}) \tag{1.6}$$

is surjective. Therefore $G'_{|\pi^{-1}(U)}$ is a direct summand of $\pi^*(V) \otimes G_{|\pi^{-1}(U)}$. □

From now on, we assume the following:

- G is a local projective generator of $\mathcal{C}$ which is a locally free sheaf.

Proposition 1.1.3. *([VB, Lem. 3.2, Cor. 3.2.8]) We set $\mathcal{A} := \pi_*(G^\vee \otimes G)$. Then we have an equivalence*

$$\begin{array}{ccc} \mathcal{C} & \to & \operatorname{Coh}_{\mathcal{A}}(Y) \\ E & \mapsto & \mathbf{R}\pi_*(G^\vee \otimes E) \end{array} \tag{1.7}$$

whose inverse is $F \mapsto \pi^{-1}(F) \overset{\mathbf{L}}{\otimes}_{\pi^{-1}(\mathcal{A})} G$. Moreover this equivalence induces an equivalence $\mathbf{D}(X) \to \mathbf{D}_{\mathcal{A}}(Y)$.

Let $\mathcal{O}_Y(1)$ be an ample line bundle on Y. For $F \in \operatorname{Coh}_{\mathcal{A}}(Y)$, we have a surjective morphism $H^0(Y, F(n)) \otimes \mathcal{A}(-n) \to F$, $n \gg 0$. Hence we have a resolution $V^\bullet \to F$ by locally free $\mathcal{A}$-modules V^i. If $V^i_{|U} \cong \mathcal{A}_U^{\oplus n}$ on an open subset of Y, then $(\pi^{-1}(V^i) \otimes_{\pi^{-1}(\mathcal{A})} G)_{|\pi^{-1}(U)} \cong G^{\oplus n}_{|\pi^{-1}(U)}$. Thus $\pi^{-1}(F) \overset{\mathbf{L}}{\otimes}_{\pi^{-1}(\mathcal{A})} G$ is isomorphic to $\pi^{-1}(V^\bullet) \otimes_{\pi^{-1}(\mathcal{A})} G$.

Assumption 1.1.4. From now on, we assume that $\dim \pi^{-1}(y) \le 1$ for all $y \in Y$ and set

$$Y_\pi := \{y \in Y \,|\, \dim \pi^{-1}(y) = 1\}. \tag{1.8}$$

Lemma 1.1.5. *Assume that* $\dim \pi^{-1}(y) \le 1$ *for all* $y \in Y$*. Let* G *be a locally free sheaf on* X *and set*

$$\begin{aligned} T :=& \{E \in \operatorname{Coh}(X) | R^1\pi_*(G^\vee \otimes E) = 0\}, \\ S :=& \{E \in \operatorname{Coh}(X) | \pi_*(G^\vee \otimes E) = 0\}. \end{aligned} \tag{1.9}$$

(1) (T,S) *is a torsion pair of* $\operatorname{Coh}(X)$ *such that* $G \in T$ *if and only if* $R^1\pi_*(G^\vee \otimes G) = 0$ *and* $S \cap T = 0$.

(2) *If* (T,S) *is a torsion pair such that* $G \in T$*, then* G *is a local projective generator of the tilted category*

$$\mathcal{C}_G := \{E \in \mathbf{D}(X) | H^{-1}(E) \in S, H^0(E) \in T,\ H^i(E) = 0,\ i \ne -1, 0\}. \tag{1.10}$$

(3) *Assume that* (T,S) *is a torsion pair such that* $G \in T$*. If* (T',S') *is a torsion pair of* $\operatorname{Coh}(X)$ *such that* $G \in T'$ *and* $S \cap T' = 0$*, then* $(T',S') = (T,S)$.

Proof. (1) We shall prove that (T,S) is a torsion pair under $R^1\pi_*(G^\vee \otimes G) = 0$ and $S \cap T = 0$. For $E \in \operatorname{Coh}(X)$, let $\phi : \pi^*(\pi_*(G^\vee \otimes E)) \otimes G \to E$ be the evaluation map. Then we see that $\pi_*(G^\vee \otimes \operatorname{coker}\phi) = 0$, $R^1\pi_*(G^\vee \otimes \operatorname{im}\phi) = 0$ and $R^1\pi_*(G^\vee \otimes E) \cong R^1\pi_*(G^\vee \otimes \operatorname{coker}\phi)$. Hence we have a desired decomposition

$$0 \to E_1 \to E \to E_2 \to 0 \tag{1.11}$$

where $E_1 := \operatorname{im}\phi \in T$ and $E_2 := \operatorname{coker}\phi \in S$.

(2) If (T,S) is a torsion pair, then for $E \in \mathcal{C}_G$, we have an exact sequence

$$0 \to R^1\pi_*(G^\vee \otimes H^{-1}(E)) \to \mathbf{R}\pi_*(G^\vee \otimes E) \to \pi_*(G^\vee \otimes H^0(E)) \to 0. \tag{1.12}$$

Hence $\mathbf{R}\pi_*(G^\vee \otimes E) \in \operatorname{Coh}(Y)$ and $\mathbf{R}\pi_*(G^\vee \otimes E) = 0$ if and only if $R^1\pi_*(G^\vee \otimes H^{-1}(E)) = \pi_*(G^\vee \otimes E) = 0$, which is equivalent to $H^{-1}(E), H^0(E) \in S \cap T = 0$. Therefore G is a local projective generator of $\mathcal{C}_G$.

(3) We first prove that $T \subset T'$. For an object $E \in T$, (2) implies that there is a surjective morphism $\phi : \pi^*(V) \otimes G \to E$ in $\mathcal{C}_G$, where V is a locally free sheaf on Y. Since ϕ is surjective in $\operatorname{Coh}(X)$ and $G \in T'$, $E \in T'$. Since $S \cap T' = 0$, we get $S \subset S'$. Therefore $(T',S') = (T,S)$. □

By the proof of Lemma 1.1.5, we get the following.

Corollary 1.1.6. *Let* G *be a locally free sheaf on* X *which gives a local projective generator of* $\mathcal{C}_G$ *in Lemma 1.1.5. Let* E *be a coherent sheaf on* X *and* $\phi : \pi^*(\pi_*(G^\vee \otimes E)) \otimes G \to E$ *the evaluation map. Then* $E_1 := \operatorname{im}\phi \in T$ *and* $E_2 := \operatorname{coker}\phi \in S$*. Thus we have a decomposition of* E

$$0 \to \operatorname{im}\phi \to E \to \operatorname{coker}\phi \to 0 \tag{1.13}$$

with respect to the torsion pair (T,S).

Lemma 1.1.7. *Assume that the local projective generator* $G \in \mathcal{C}$ *is a locally free sheaf. We set*

$$\begin{aligned} T :=& \{E \in \operatorname{Coh}(X) | R^1\pi_*(G^\vee \otimes E) = 0\}, \\ S :=& \{E \in \operatorname{Coh}(X) | \pi_*(G^\vee \otimes E) = 0\}. \end{aligned} \tag{1.14}$$

Then (T,S) *is a torsion pair of* $\operatorname{Coh}(X)$ *whose tilting is* $\mathcal{C}$.

Proof. Since $G \in \mathcal{C}$, we have $\mathbf{R}\pi_*(G^\vee \otimes G) \in \operatorname{Coh}(Y)$. By the definition of a local projective generator, we have $S \cap T = 0$. By Lemma 1.1.5, (T,S) is a torsion pair. Let $\mathcal{C}_G$ be the tilted category. Since $S[1], T \subset \mathcal{C}$, we get $\mathcal{C}_G \subset \mathcal{C}$. Conversely for $E \in \mathcal{C}$, we have a spectral sequence

$$E_2^{p,q} = R^p\pi_*(G^\vee \otimes H^q(E)) \Longrightarrow E_\infty^{p+q} = R^{p+q}\pi_*(G^\vee \otimes E). \tag{1.15}$$

Since $\pi^{-1}(y) \le 1$ for all $y \in Y$, this spectral sequence degenerates. Hence we have $\mathbf{R}\pi_*(G^\vee \otimes H^q(E)) = 0$ for $q \ne -1, 0$, $\pi_*(G^\vee \otimes H^{-1}(E)) = 0$ and $R^1\pi_*(G^\vee \otimes H^0(E)) = 0$. Therefore $E \in \mathcal{C}_G$. □

Lemma 1.1.8. *For the locally free sheaf* G *on* X *and the tilted category* $\mathcal{C}_G$ *in Lemma 1.1.5, we set*

$$\begin{aligned} T^D :=& \{E \in \operatorname{Coh}(X) | R^1\pi_*(G \otimes E) = 0\}, \\ S^D :=& \{E \in \operatorname{Coh}(X) | \pi_*(G \otimes E) = 0\}. \end{aligned} \tag{1.16}$$

Then (T^D, S^D) *is a torsion pair and* $G^\vee$ *is a local projective generator of the tilted category. We denote the tilted category by* $\mathcal{C}_G^D$.

Proof. Since $R^1\pi_*(G^\vee \otimes G) = 0$, $G^\vee \in T^D$. We show that $T^D \cap S^D = 0$. Assume that $\mathbf{R}\pi_*(G \otimes E) = 0$ for a coherent sheaf E on X. Since

$$\begin{aligned} H^i(Y, \mathbf{R}\pi_*(G \otimes E)(-k)) =& H^i(X, G \otimes E(-k)) \\ =& H^{n-i}(X, G^\vee \otimes D_X(E)(K_X) \otimes \mathcal{O}_X(k))^\vee \\ =& H^{n-i}(Y, \mathbf{R}\pi_*(G^\vee \otimes D_X(E)(K_X))(k))^\vee \end{aligned} \tag{1.17}$$

for all $k \in \mathbb{Z}$ and $H^j(Y, H^{n-i}(\mathbf{R}\pi_*(G^\vee \otimes D_X(E)(K_X)))(k)) = 0$ for $k \gg 0$ and $j \ne 0$, we get $H^{n-i}(Y, \mathbf{R}\pi_*(G^\vee \otimes D_X(E)(K_X))(k)) = H^0(Y, H^{n-i}(\mathbf{R}\pi_*(G^\vee \otimes D_X(E)(K_X)))(k)) = 0$ for $k \gg 0$. Therefore $\mathbf{R}\pi_*(G^\vee \otimes$

$D_X(E)(K_X)) = 0$. Since $\dim \pi^{-1}(y) \leq 1$ for all $y \in Y$, we see that $\mathbf{R}\pi_*(G^\vee \otimes H^i(D_X(E)(K_X))) = \mathbf{R}\pi_*(H^i(G^\vee \otimes D(E)(K_X))) = 0$ (see the proof of Lemma 1.1.7). Since G is a local projective generator of $\mathcal{C}_G$, $H^i(D_X(E)(K_X)) = 0$ for all i. Therefore $D_X(E)(K_X) = 0$, which implies that $E = 0$. □

Remark 1.1.9. If E is a local projective object of $\mathcal{C}_G$, that is, $R^1\pi_*(E^\vee \otimes F) = 0$ for all $F \in \mathcal{C}_G$, then $E^\vee \in \mathcal{C}_G^D$. Indeed by $G \in \mathcal{C}_G$, we have $R^1\pi_*(E^\vee \otimes G) = 0$, which implies that $E^\vee \in T^D$. Moreover since $G^\vee$ is a local projective generator of $\mathcal{C}^D$ and $R^1\pi_*(E \otimes G^\vee) = 0$, $E^\vee$ is a local projective object of $\mathcal{C}^D$.

1.1.1. *Irreducible objects of $\mathcal{C}$.*

Lemma 1.1.10. *Let G be a locally free sheaf on X such that $\mathbf{R}\pi_*(G^\vee \otimes F) \neq 0$ for all non-zero coherent sheaf F on a fiber of π. Then for a coherent sheaf E on X, $\pi_*(G^\vee \otimes E) = 0$ implies $R^1\pi_*(G^\vee \otimes E_{|\pi^{-1}(y)}) \neq 0$ for all $y \in \pi(\operatorname{Supp}(E))$.*

Proof. Assume that $R^1\pi_*(G^\vee \otimes E_{|\pi^{-1}(y)}) = 0$. By Lemma 1.1.16 below, $R^1\pi_*(G^\vee \otimes E) = 0$ in a neighborhood of y. Thus $\mathbf{R}\pi_*(G^\vee \otimes E) = 0$ in a neighborhood of y. Then $\mathbf{R}\pi_*(G^\vee \otimes E \overset{\mathbf{L}}{\otimes} \mathbf{L}\pi^*(\mathbb{C}_y)) = \mathbf{R}\pi_*(G^\vee \otimes E) \overset{\mathbf{L}}{\otimes} \mathbb{C}_y = 0$. Since the spectral sequence

$$E_2^{pq} = R^p\pi_*(H^q(G^\vee \otimes E \overset{\mathbf{L}}{\otimes} \mathbf{L}\pi^*(\mathbb{C}_y))) \Longrightarrow E_\infty^{p+q} = H^{p+q}(\mathbf{R}\pi_*(G^\vee \otimes E \overset{\mathbf{L}}{\otimes} \mathbf{L}\pi^*(\mathbb{C}_y))) \tag{1.18}$$

degenerates, $R^p\pi_*(G^\vee \otimes E \otimes \pi^*(\mathbb{C}_y)) = 0$. By our assumption on G, we have $E_{|\pi^{-1}(y)} = 0$, which is a contradiction. □

Definition 1.1.11.
(1) An object $E \in \mathcal{C}$ is 0-*dimensional*, if $\mathbf{R}\pi_*(G^\vee \otimes E)$ is 0-dimensional as an object of $\operatorname{Coh}(Y)$.
(2) An object $E \in \mathcal{C}$ is *irreducible*, if E does not have a proper subobject except 0.
(3) For a 0-dimensional object $E \in \mathcal{C}$, we take a filtration

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E \tag{1.19}$$

such that F_i/F_{i-1} are irreducible objects of $\mathcal{C}$. Then $\oplus_i F_i/F_{i-1}$ is the *Jordan-Hölder decomposition* of E.

Remark 1.1.12. In section 1.4, we shall define the dimension of E generally. According to the definition of the stability in Definition 1.4.1, we also have the following.
(1) A 0-dimensional object E is G-twisted semi-stable and a G-twisted stable object corresponds to an irreducible object.
(2) The Jordan-Hölder decomposition of E is nothing but the standard representative of the S-equivalence class of E.

Lemma 1.1.13. *Let G be a locally free sheaf on X and $\mathcal{C}_G$ the tilted category in Lemma 1.1.5.*
(1) *$\mathbb{C}_x \in \mathcal{C}_G$ for all $x \in X$.*
(2) *For $\mathbb{C}_x, x \in \pi^{-1}(y)$, the Jordan-Hölder decomposition does not depend on the choice of $x \in \pi^{-1}(y)$.*
(3) *Let $\oplus_{j=0}^{s_y} E_{yj}^{\oplus a_{yj}}$ be the Jordan-Hölder decomposition of $\mathbb{C}_x$, $x \in \pi^{-1}(y)$. Then the irreducible objects of $\mathcal{C}_G$ are*

$$\mathbb{C}_x, (x \in X \setminus \pi^{-1}(Y_\pi)), \quad E_{yj}, (y \in Y_\pi,\ 0 \leq j \leq s_y). \tag{1.20}$$

In particular, if $\mathbf{R}\pi_*(G^\vee \otimes E)$ is a 0-dimensional $\mathcal{A}$-module, then E is generated by (1.20).

Proof. (1) We note that $\mathbf{R}\pi_*(G^\vee \otimes \mathbb{C}_x) = \pi_*(G^\vee \otimes \mathbb{C}_x)$. Hence $\mathbb{C}_x \in \mathcal{C}_G$. (2) Since the trace map $G^\vee \otimes G \to \mathcal{O}_X$ is surjective, we have a surjective map

$$R^1\pi_*(G^\vee \otimes G) \to R^1\pi_*(\mathcal{O}_X) \to R^1\pi_*(\mathcal{O}_{\pi^{-1}(y)_{\mathrm{red}}}), \tag{1.21}$$

where $\pi^{-1}(y)_{\mathrm{red}}$ is the reduced subscheme of $\pi^{-1}(y)$. Since $R^1\pi_*(G^\vee \otimes G) = 0$, we get

$$H^1(\pi^{-1}(y)_{\mathrm{red}}, \mathcal{O}_{\pi^{-1}(y)_{\mathrm{red}}}) = H^0(Y, R^1\pi_*(\mathcal{O}_{\pi^{-1}(y)_{\mathrm{red}}})) = 0.$$

Then we see that $\pi^{-1}(y)_{\mathrm{red}}$ is a tree of smooth rational curves. Let C_{yj}, $j = 0, ..., s_y$ be the irreducible component of $\pi^{-1}(y)_{\mathrm{red}}$. Since the restriction map $R^1\pi_*(G^\vee \otimes G) \to R^1\pi_*(G^\vee \otimes G_{|C_{yj}})$ is surjective, $R^1\pi_*(G^\vee \otimes G_{|C_{yj}}) = 0$. Thus we can write $G_{|C_{yj}} \cong \mathcal{O}_{C_{yj}}(d_{yj})^{\oplus r_{yj}} \oplus \mathcal{O}_{C_{yj}}(d_{yj}+1)^{\oplus r'_{yj}}$. Since $R^1\pi_*(G^\vee \otimes \mathcal{O}_{C_{yj}}(d_{yj})) = 0$ and $\pi_*(G^\vee \otimes \mathcal{O}_{C_{yj}}(d_{yj}-1)) = 0$, $\mathcal{O}_{C_{yj}}(d_{yj}), \mathcal{O}_{C_{yj}}(d_{yj}-1)[1] \in \mathcal{C}_G$. For $x \in C_{yj}$, we have an exact sequence in $\mathcal{C}_G$

$$0 \to \mathcal{O}_{C_{yj}}(d_{yj}) \to \mathbb{C}_x \to \mathcal{O}_{C_{yj}}(d_{yj}-1)[1] \to 0. \tag{1.22}$$

Hence the Jordan-Hölder decomposition of $\mathbb{C}_x$ is constant on C_{yj}. Since $\pi^{-1}(y)$ is connected, the Jordan-Hölder decomposition of $\mathbb{C}_x$ is determined by y.

(3) Let E be an irreducible object of $\mathcal{C}_G$. Then we have (i) $E = F[1], F \in \mathrm{Coh}(X)$ or (ii) $E \in \mathrm{Coh}(X)$. In the first case, since $F \in S$, we have $\pi_*(G^\vee \otimes F) = 0$. By Lemma 1.1.10, we have $R^1\pi_*(G^\vee \otimes F_{|\pi^{-1}(y)}) \neq 0$ for $y \in \pi(\mathrm{Supp}(F))$, which implies that there is a quotient $F_{|\pi^{-1}(y)} \to F'$ such that $0 \neq F' \in S$ for $y \in \pi(\mathrm{Supp}(F))$. Then we have a non-trivial morphism $F[1] \to F'[1]$, which should be injective in $\mathcal{C}_G$. Therefore $\pi(\mathrm{Supp}(F))$ is a point. In the second case, we also see that $\pi(\mathrm{Supp}(E))$ is a point. Therefore $\mathbf{R}\pi_*(G^\vee \otimes E)$ is a 0-dimensional sheaf. (i) If $E = F[1]$, then since $\pi_*(G^\vee \otimes F) = 0$, F is purely 1-dimensional. Then $\mathrm{Hom}(\mathbb{C}_x, F[1]) = \mathrm{Hom}(D(F)[n-1], D(\mathbb{C}_x)[n]) \neq 0$ for $x \in \mathrm{Supp}(F)$, where $n = \dim X$. Hence we have a non-trivial morphism $E_{yj} \to E$, $y \in \pi(\mathrm{Supp}(F)) \cap Y_\pi$, which is an isomorphism. (ii) If $E \in \mathrm{Coh}(X)$, then $\mathrm{Hom}(E, \mathbb{C}_x) \neq 0$ for $x \in \mathrm{Supp}(E)$, which also implies that $E \cong E_{yj}$ for $\mathrm{Supp}(E) \subset \pi^{-1}(y)$ or $E \cong \mathbb{C}_x$ for $\mathrm{Supp}(E) \subset X \setminus Y_\pi$. □

Remark 1.1.14. Since $\pi_*(G^\vee \otimes \mathbb{C}_x)$ is a coherent sheaf on the reduced point $\{y\}$, the multiplication $\pi^*(t) : E_{yj} \to E_{yj}$, $t \in I_y$ is zero. Thus $H^i(E_{yj})$ are coherent sheaves on the scheme $\pi^{-1}(y)$.

Lemma 1.1.15. *Let E be a coherent sheaf such that $\pi(\mathrm{Supp}(E)) = \{y\}$.*

(1) *For $E \in T$ with $\mathrm{Supp}(E) \subset \pi^{-1}(y)$, there is a filtration*

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E \tag{1.23}$$

such that for every F_k/F_{k-1}, there is $E_{yj} \in T$ and a surjective homomorphism $E_{yj} \to F_k/F_{k-1}$ in $\mathrm{Coh}(X)$.

(2) *For $E \in S$, there is a filtration*

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E \tag{1.24}$$

such that for every F_k/F_{k-1}, there is $E_{yj}[-1] \in S$ and an injective homomorphism $F_k/F_{k-1} \to E_{yj}[-1]$ in $\mathrm{Coh}(X)$.

Proof. (1) Since $E \in T$, E contains E_{yj} in $\mathcal{C}$. Let F be the quotient in $\mathcal{C}$. Then we have an exact sequence

$$0 \to H^{-1}(E_{yj}) \to 0 \to H^{-1}(F) \to H^0(E_{yj}) \to E \to H^0(F) \to 0. \tag{1.25}$$

Hence $E_{yj} \in T$ and $H^0(F) \in T$. We set $F_1 := \mathrm{im}(E_{yj} \to E)$ in $\mathrm{Coh}(X)$. Since $E/F_1 \in T$ and $\mathrm{Supp}(E/F_1) \subset \pi^{-1}(y)$, by the induction on the support of E, we get the claim.

(2) Since $E \in S$, there is a quotient $E[1] \to E_{yj}$ in $\mathcal{C}$. Let F be the kernel in $\mathcal{C}$. Then we have an exact sequence

$$0 \to H^{-1}(F) \to E \to H^{-1}(E_{yj}) \to H^0(F) \to 0 \to H^0(E_{yj}) \to 0. \tag{1.26}$$

Hence $E_{yj}[-1] \in S$ and $H^{-1}(F) \in S$. We set $E' := \mathrm{im}(E \to H^{-1}(E_{yj}))$ in $\mathrm{Coh}(X)$. Then E' is a subsheaf of $E_{yj}[-1]$ and E is an extension of E' by $H^{-1}(F) \in S$. Since $\mathrm{Supp}(H^{-1}(F)) \subset \pi^{-1}(y)$, by the induction on the support of E, we get the claim. □

Lemma 1.1.16. (1) *$\pi^*(\pi_*(I_{\pi^{-1}(y)})) \to I_{\pi^{-1}(y)}$ is surjective. In particular, $\mathrm{Hom}(I_{\pi^{-1}(y)}, \mathcal{O}_{C_{yj}}(-1)) = 0$ for all j.*

(2) *$\mathrm{Ext}^1(\mathcal{O}_{\pi^{-1}(y)}, \mathcal{O}_{C_{yj}}(-1)) = 0$ for all j. In particular,*

$$H^1(X, \mathcal{H}om(\mathcal{O}_{\pi^{-1}(y)}, \mathcal{O}_{C_{yj}}(-1))) = H^0(X, \mathcal{E}xt^1(\mathcal{O}_{\pi^{-1}(y)}, \mathcal{O}_{C_{yj}}(-1))) = 0.$$

(3) *For a coherent sheaf E on X, $R^1\pi_*(E) = 0$ at y if and only if $R^1\pi_*(E_{|\pi^{-1}(y)}) = 0$.*

Proof. Since $I_{\pi^{-1}(y)} = \mathrm{im}(\pi^*(I_y) \to \mathcal{O}_X)$, (1) holds. (2) Since $\mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_{C_{yj}}(-1)[k]) = 0$ for all j and k, the first claim follows from the exact sequence

$$0 \to I_{\pi^{-1}(y)} \to \mathcal{O}_X \to \mathcal{O}_{\pi^{-1}(y)} \to 0. \tag{1.27}$$

Since $H^2(X, \mathcal{H}om(\mathcal{O}_{\pi^{-1}(y)}, \mathcal{O}_{C_{yj}}(-1))) = 0$, the second claim follows from the local-global spectral sequence.

(3) The proof is similar to [Is1]. Assume that $R^1\pi_*(E_{|\pi^{-1}(y)}) = 0$. We take a locally free sheaf V on Y such that $V \to I_y$ is surjective. Then (1) implies that $\pi^*(V) \to I_{\pi^{-1}(y)}$ is surjective. Hence we have a surjective homomorphism $\pi^*(V^{\otimes n}) \otimes \mathcal{O}_{\pi^{-1}(y)} \to I^n_{\pi^{-1}(y)}/I^{n+1}_{\pi^{-1}(y)}$. Then we see that $R^1\pi_*(E \otimes \mathcal{O}_X/I^n_{\pi^{-1}(y)}) = 0$. By the theorem of formal functions, we get the claim. □

Lemma 1.1.17. *Let E_{yj}, $y \in Y_\pi$ be the irreducible objects of $\mathcal{C}_G$. Let E be a coherent sheaf on X. If $\mathrm{Hom}(E, E_{yj}[-1]) = 0$ for all $E_{yj}[-1] \in S$, then $E \in T$.*

Proof. We note that $\mathrm{Hom}(E_{|\pi^{-1}(y)}, E_{yj}[-1]) = 0$ for all $E_{yj}[-1] \in S$. By Lemma 1.1.15 (2), $E_{|\pi^{-1}(y)} \in T$. Then $R^1\pi_*(G^\vee \otimes E_{|\pi^{-1}(y)}) = 0$. By Lemma 1.1.16, $R^1\pi_*(G^\vee \otimes E) = 0$ in a neighborhood of y. Since y is any point of Y_π, $R^1\pi_*(G^\vee \otimes E) = 0$, which implies that $E \in T$. □

For a subcategory $\mathcal{C}$ of $\mathbf{D}(X)$, we set

$$\mathcal{C}_y := \{E \in \mathcal{C} | \pi(\mathrm{Supp}(H^i(E))) = \{y\}, i \in \mathbb{Z}\}. \tag{1.28}$$

Lemma 1.1.18. *Let (S,T) be a torsion pair of $\operatorname{Coh}(X)$ and $\mathcal{C}$ the tilted category. Assume that*

(i) *$\#Y_\pi < \infty$ and every object of $\mathcal{C}_y$, $y \in Y$ is of finite length.*
(ii) *$\mathbb{C}_x \in \mathcal{C}$ for all $x \in X$.*
(iii) *$\pi(\operatorname{Supp}(E)) \subset Y_\pi$ for $E \in S$.*

Then the claims of Lemma 1.1.15 and Lemma 1.1.17 hold.

Proof. By (i) and (iii), irreducible objects are $E = \mathbb{C}_x, x \in X \setminus \pi^{-1}(Y_\pi)$ or irreducible objects of $\mathcal{C}_y, y \in Y_\pi$. By (ii), we get Lemma 1.1.13 (3). The other claims of Lemma 1.1.13 and Lemma 1.1.15 are obvious. For $0 \neq E \in S$, (i) and Lemma 1.1.15 imply that there is a coherent sheaf $E_{yj}[-1] \in S$ such that $\operatorname{Hom}(E, E_{yj}[-1]) \neq 0$. Hence Lemma 1.1.17 also holds. □

Proposition 1.1.19. *Assume that $Y_\pi = \{p_1, ..., p_m\}$. Let G be a locally free sheaf on X and $\mathcal{C}_G$ the tilted category in Lemma 1.1.5. For $\mathbb{C}_x, x \in \pi^{-1}(p_i)$, let $\oplus_{j=0}^{s_i} E_{ij}^{\oplus a_{ij}}$ be the Jordan-Hölder decomposition of $\mathbb{C}_x$, where E_{ij} are irreducible objects.*

(1) *We set*

$$\begin{aligned}\Sigma :=& \{E_{ij}[-1] | i,j\} \cap \operatorname{Coh}(X)\\ \mathcal{T} :=& \{E \in \operatorname{Coh}(X) | \operatorname{Hom}(E,c) = 0, c \in \Sigma\}\\ \mathcal{S} :=& \{E \in \operatorname{Coh}(X) | \ E \text{ is a successive extension of subsheaves of } c \in \Sigma\ \}.\end{aligned} \tag{1.29}$$

Then $(\mathcal{T}, \mathcal{S})$ is a torsion pair of $\operatorname{Coh}(X)$ whose tilting is $\mathcal{C}_G$. In particular, $\mathcal{C}_G$ is characterized by Σ.

(2) *$\mathcal{C}_G^D$ is characterized by*

$$\Sigma := \{(D_X(E_{ij}) \otimes K_X[n])[-1] | i,j\} \cap \operatorname{Coh}(X) = D_X(\{E_{ij} | i,j\} \cap \operatorname{Coh}(X)) \otimes K_X[n-1], \tag{1.30}$$

where $n = \dim X$.

Proof. (1) For $E \in \operatorname{Coh}(X)$, we consider $\phi : G \otimes \pi^*(\pi_*(G^\vee \otimes E)) \to E$. We set $E_1 := \operatorname{im} \phi$ and $E_2 := \operatorname{coker} \phi$. Since $\operatorname{Hom}(G, E_{ij}[-1]) = 0$ for all E_{ij}, $G \in \mathcal{T}$. Hence $E_1 \in \mathcal{T}$. We shall show that $E_2 \in \mathcal{S}$. We note that $\mathbf{R}\pi_*(G^\vee \otimes E_1) = \pi_*(G^\vee \otimes E_1)$ and $\mathbf{R}\pi_*(G^\vee \otimes E_2) = R^1\pi_*(G^\vee \otimes E)[-1]$. Then $E_1, E_2[1] \in \mathcal{C}_G$. Since $\operatorname{Supp}(E_2) \subset \cup_{i=1}^n \pi^{-1}(p_i)$, Lemma 1.1.13 (3) implies that $E_2[1]$ is generated by E_{ij}. Hence if $E_2 \neq 0$, then $\operatorname{Hom}(E_2[1], c[1]) \neq 0$ for an object $c \in \Sigma$. Let E_2' be the kernel of $E_2 \to c$ in $\operatorname{Coh}(X)$. Then $E_2'[1] \in \mathcal{C}_G$. Hence by the induction on the support of E_2, we see that $E_2 \in \mathcal{S}$. Therefore $(\mathcal{T}, \mathcal{S})$ is a torsion pair of $\operatorname{Coh}(X)$. We also see that

$$\begin{aligned}\mathcal{T} &= \{E \in \operatorname{Coh}(X) | R^1\pi_*(G^\vee \otimes E) = 0\},\\ \mathcal{S} &= \{E \in \operatorname{Coh}(X) | \pi_*(G^\vee \otimes E) = 0\}\end{aligned} \tag{1.31}$$

and $\mathcal{C}_G$ is the tilting of $\operatorname{Coh}(X)$.

(2) We note that $\mathbb{C}_x, x \in \pi^{-1}(p_i)$ is S-equivalent to $\oplus_{j=0}^{s_i} D_X(E_{ij}) \otimes K_X[n]^{\oplus a_{ij}}$, where $D_X(E_{ij}) \otimes K_X[n] \in \mathcal{C}_G^D$. Hence the claim follows from (1). □

1.1.2. *Local projective generators of $\mathcal{C}$.* Let (S,T) be a torsion pair of $\operatorname{Coh}(X)$ such that the tilted category $\mathcal{C}$ satisfies one of the following conditions.

(1) There is a local projective generator $G \in T$ of $\mathcal{C}$, that is, $\mathcal{C}$ is the category of perverse coherent sheaves or
(2) $\mathcal{C}$ satisfies the following conditions:
   (a) $\#Y_\pi < \infty$ and every object of $\mathcal{C}_y$, $y \in Y$ is of finite length.
   (b) $\pi(\operatorname{Supp}(E)) \subset Y_\pi$ for $E \in S$.

We shall give a criterion for a two term complex to be a local projective generator of $\mathcal{C}$. Let $E_{yj}, j \in J_y$ be the irreducible objects of $\mathcal{C}_y$.

Lemma 1.1.20. *Let E be an object of $\mathbf{D}(X)$ such that $H^i(E) = 0$ for $i \neq -1, 0$. If $\operatorname{Ext}^1(E, \mathbb{C}_x) = 0$, then E is a free sheaf in a neighborhood of x.*

Proof. Since E fits in the exact triangle

$$0 \to H^{-1}(E)[1] \to E \to H^0(E) \to H^{-1}(E)[2], \tag{1.32}$$

we have an exact sequence

$$0 \to \mathcal{E}xt^1_{\mathcal{O}_X}(H^0(E), \mathbb{C}_x) \to \mathcal{E}xt^1_{\mathcal{O}_X}(E, \mathbb{C}_x) \to \mathcal{H}om_{\mathcal{O}_X}(H^{-1}(E), \mathbb{C}_x) \to \mathcal{E}xt^2_{\mathcal{O}_X}(H^0(E), \mathbb{C}_x). \tag{1.33}$$

Since $\operatorname{Ext}^1(E, \mathbb{C}_x) = H^0(X, \mathcal{E}xt^1_{\mathcal{O}_X}(E, \mathbb{C}_x))$, $\mathcal{E}xt^1_{\mathcal{O}_X}(E, \mathbb{C}_x) = 0$. Then $\mathcal{E}xt^1_{\mathcal{O}_X}(H^0(E), \mathbb{C}_x) = 0$, which implies that $H^0(E)$ is a free sheaf in a neighborhood of x. Then $\mathcal{E}xt^i_{\mathcal{O}_X}(H^0(E), \mathbb{C}_x) = 0$ for $i > 0$. Hence $\mathcal{H}om_{\mathcal{O}_X}(H^{-1}(E), \mathbb{C}_x) = 0$. Therefore $H^{-1}(E) = 0$ in a neighborhood of x. □

Lemma 1.1.21. *Let E_{yj}, $y \in Y$ be the irreducible objects of $\mathcal{C}$ in Lemma 1.1.13. Let G_1 be a locally free sheaf of rank r on X such that*

$$(a)\ \operatorname{Hom}(G_1, E_{yj}[p]) = 0, p \neq 0 \quad (b)\ \chi(G_1, E_{yj}) > 0 \tag{1.34}$$

for all y, j.

1. *G_1 is a locally free sheaf. If $0 \neq E \in S$, then $\pi_*(G_1^\vee \otimes E) = 0$ and $R^1\pi_*(G_1^\vee \otimes E) \neq 0$.*
2. *If $R^1\pi_*(G_1^\vee \otimes E) = 0$, then $E \in T$.*
3. *If $0 \neq E \in T$ and $\operatorname{Supp}(E) \subset \pi^{-1}(y)$, then $\pi_*(G_1^\vee \otimes E) \neq 0$ and $R^1\pi_*(G_1^\vee \otimes E) = 0$. In particular, $\chi(G_1, E) > 0$.*

Proof. (1) We note that $G_1 \in T$ by Lemma 1.1.17. We first treat the case where $\mathcal{C}$ is the category of perverse coherent sheaves. We consider the homomorphism $\pi^*(\pi_*(G_1^\vee \otimes E)) \otimes G_1 \to E$. Then $\operatorname{im}\phi \in T \cap S = 0$. Since $\pi_*(G_1^\vee \otimes \operatorname{im}\phi) = \pi_*(G_1^\vee \otimes E)$, we get $\pi_*(G_1^\vee \otimes E) = 0$. Let $F \neq 0$ be a coherent sheaf on a fiber and take the decomposition

$$0 \to F_1 \to F \to F_2 \to 0 \tag{1.35}$$

with $F_1 \in T, F_2 \in S$. Since $F_1, F_2[1] \in \mathcal{C}$, the condition $\chi(G_1, E_{yj}) > 0$ implies that $\chi(G_1, F_1) > 0$ or $\chi(G_1, F_2) < 0$, which imply that $\pi_*(G_1^\vee \otimes F_1) \neq 0$ or $R^1\pi_*(G_1^\vee \otimes F_2) \neq 0$. Since $\pi_*(G_1^\vee \otimes F_1)$ is a subsheaf of $\pi_*(G_1^\vee \otimes F)$ and $R^1\pi_*(G_1^\vee \otimes F_2)$ is a quotient of $R^1\pi_*(G_1^\vee \otimes F)$, we get $\mathbf{R}\pi_*(G_1^\vee \otimes F) \neq 0$. Then we can apply Lemma 1.1.10 to E and get $R^1\pi_*(G_1^\vee \otimes E_{|\pi^{-1}(y)}) \neq 0$ for $y \in \pi(\operatorname{Supp}(E))$. Since $R^1\pi_*(G_1^\vee \otimes E) \to R^1\pi_*(G_1^\vee \otimes E_{|\pi^{-1}(y)}) \neq 0$ is surjective, we get the claim.

We next assume that $\#Y_\pi < \infty$. Then $E[1]$ is generated by E_{yj}. Hence (1.34) imply that $\chi(G_1, E[1]) > 0$ and $\mathbf{R}\pi_*(G_1^\vee \otimes E[1]) \in \operatorname{Coh}(Y)$. Hence $R^1\pi_*(G_1^\vee \otimes E) \neq 0$ and $\pi_*(G_1^\vee \otimes E) = 0$.

(2) For $E \in \operatorname{Coh}(X)$, we take a decomposition

$$0 \to E_1 \to E \to E_2 \to 0 \tag{1.36}$$

such that $E_1 \in T$ and $E_2 \in S$. If $R^1\pi_*(G_1^\vee \otimes E) = 0$, then (1) implies that $E_2 = 0$.

(3) By Lemma 1.1.15, we may assume that E is a quotient of E_{yj}, $E_{yj} \in T$ in $\operatorname{Coh}(X)$. Since E_{yj} is irreducible, $\phi : E_{yj} \to E$ is injective in $\mathcal{C}$. We set $F := \ker(E_{yj} \to E)$ in $\operatorname{Coh}(X)$. Then $F \in S$ and $F[1]$ is the cokernel of ϕ in $\mathcal{C}$. Hence $\pi_*(G_1^\vee \otimes F) = 0$ by (1). By our assumption, $\pi_*(G_1^\vee \otimes E_{yj}) \neq 0$, $E_{yj} \in T$ and $R^1\pi_*(G_1^\vee \otimes E_{yj}) = 0$. Therefore our claim holds. □

Proposition 1.1.22. *Let G_1 be an object of $\mathbf{D}(X)$ such that $H^i(E) = 0$ for $i \neq -1, 0$ and satisfies*

$$(a)\ \operatorname{Hom}(G_1, E_{yj}[p]) = 0, p \neq 0 \quad (b)\ \chi(G_1, E_{yj}) > 0. \tag{1.37}$$

1. *G_1 is a locally free sheaf on X.*
2. *$R^1\pi_*(G_1^\vee \otimes G_1) = 0$.*
3. *For $E \in \operatorname{Coh}(X)$, $E \in T$ if and only if $R^1\pi_*(G_1^\vee \otimes E) = 0$, and $E \in S$ if and only if $\pi_*(G_1^\vee \otimes E) = 0$.*
4. *G_1 is a local projective generator of $\mathcal{C}_G$.*

Proof. (1) The claim follows from Lemma 1.1.20 and (a). (2) It is sufficient to prove that $R^1\pi_*(G_1^\vee \otimes G_{1|\pi^{-1}(y)}) = 0$ for all $y \in Y_\pi$. By Lemma 1.1.17, $G_1 \in T$. Since $\operatorname{Supp}(G_{1|\pi^{-1}(y)}) = \pi^{-1}(y)$ and $G_{1|\pi^{-1}(y)} \in T$, Lemma 1.1.15 (1) implies that $G_{1|\pi^{-1}(y)} \in T$ is a successive extension of quotients of $E_{yj} \in T$. Hence it is sufficient to prove $R^1\pi_*(G_1^\vee \otimes Q) = 0$ for all quotients Q of $E_{yj} \in T$. By our assumption on G_1, we have $R^1\pi_*(G_1^\vee \otimes E_{yj}) = 0$ for $E_{yj} \in T$. Therefore the claim holds.

(3) We set

$$\begin{aligned} T_1 :=& \{E \in \operatorname{Coh}(X) | R^1\pi_*(G_1^\vee \otimes E) = 0\}, \\ S_1 :=& \{E \in \operatorname{Coh}(X) | \pi_*(G_1^\vee \otimes E) = 0\}. \end{aligned} \tag{1.38}$$

By Lemma 1.1.21 (2), we get

$$T_1 \cap S_1 \subset T \cap S_1 = \{E \in T | \pi_*(G_1^\vee \otimes E) = 0\}. \tag{1.39}$$

If $T \cap S_1 = 0$, then Lemma 1.1.5 (1) implies that G_1 is a local projective generator of $\mathcal{C}_{G_1}$. Since $G_1 \in T$ by (2), Lemma 1.1.5 (3) also implies that $\mathcal{C} = \mathcal{C}_{G_1}$. Therefore we shall prove that $T \cap S_1 = 0$. Assume that $E \in T$ satisfies $\pi_*(G_1^\vee \otimes E) = 0$. We first prove that $R^1\pi_*(G_1^\vee \otimes E) = 0$. By Lemma 1.1.16, it is sufficient to prove $R^1\pi_*(G_1^\vee \otimes E_{|\pi^{-1}(y)}) = 0$ for all $y \in Y$. This follows from Lemma 1.1.21 (3). Hence $\mathbf{R}\pi_*(G_1^\vee \otimes E) = 0$. Then we see that $\mathbf{R}\pi_*(G_1^\vee \otimes E_{|\pi^{-1}(y)}) = 0$ for all $y \in Y$ by the proof of Lemma 1.1.10. Since $E_{|\pi^{-1}(y)} \in T$, Lemma 1.1.21 (3) implies that $E_{|\pi^{-1}(y)} = 0$ for all $y \in Y$. Therefore $E = 0$.

(4) This is a consequence of (3) and Lemma 1.1.5 (2). □

Remark 1.1.23. If G_1 in Proposition 1.1.22 satisfies (1.37) (a) only, then the proofs of Lemma 1.1.21 and Proposition 1.1.22 imply that G_1 is a locally free sheaf such that $R^1\pi_*(G_1^\vee \otimes G_1) = 0$ and $\mathbf{R}\pi_*(G_1^\vee \otimes F) \in \operatorname{Coh}(Y)$ for $F \in \mathcal{C}_G$.

Lemma 1.1.24. *Let (S,T) be a torsion pair of $\operatorname{Coh}(X)$ and $\mathcal{C}$ its tilting. Assume that one of the following holds.*

(i) *$\mathcal{C}$ is the category of perverse coherent sheaves.*
(ii) *$\#Y_\pi < \infty$, $\mathcal{C}_y$ is Artinian and $\pi(\operatorname{Supp}(E)) \subset Y_\pi$ for $E \in S$.*

Let G_1 be a locally free sheaf of rank r on X such that

$$\chi(G_1, E_{yj}) > 0. \tag{1.40}$$

Then $\operatorname{Hom}(G_1, E_{yj}[k]) = 0, k \neq 0$ if and only if $R^1\pi_*(G_1^\vee \otimes G_1) = 0$.

Proof. Assume that $R^1\pi_*(G_1^\vee \otimes G_1) = 0$. We first prove that $G_1 \in T$. Assume that $G_1 \notin T$. Then there is a surjective homomorphism $G_1 \to E$ in $\operatorname{Coh}(X)$ such that $E \in S$. If $\mathcal{C}$ has a local projective generator G, then $\pi_*(G^\vee \otimes E) = 0$. By Lemma 1.1.10, we have $R^1\pi_*(G^\vee \otimes E_{|\pi^{-1}(y)}) \neq 0$ for a point $y \in Y$. Hence we may assume that $\operatorname{Supp}(E) \subset \pi^{-1}(y)$. In the second case, since $\#Y_\pi < \infty$, we may also assume that $\operatorname{Supp}(E) \subset \pi^{-1}(y)$. Then $E[1]$ is generated by E_{yj}, $0 \le j \le s_y$. By our assumption, $\chi(G_1, E[1]) > 0$. Hence $\operatorname{Ext}^1(G_1, E) \neq 0$, which implies that $R^1\pi_*(G_1^\vee \otimes G_1) \neq 0$. Therefore $G_1 \in T$. For $E_{yj} \in T$, we consider the homomorphism $\phi : \pi^*(\pi_*(G_1^\vee \otimes E_{yj})) \otimes G_1 \to E_{yj}$. Since E_{yj} is an irreducible object, ϕ is surjective in $\mathcal{C}_G$, which implies that ϕ is surjective in $\operatorname{Coh}(X)$. Hence $\operatorname{Ext}^1(G_1, E_{yj}) = 0$. For $E_{yj} \in S[1]$, $\dim \pi^{-1}(y) \le 1$ and the locally freeness of G_1 imply that $\operatorname{Ext}^1(G_1, E_{yj}) = 0$. Since $G_1 \in T$, we also get $\operatorname{Hom}(G_1, E_{yj}[-1]) = 0$ for all irreducible objects of $\mathcal{C}$. □

Lemma 1.1.25. *Let G be a locally free sheaf on X such that $R^1\pi_*(G^\vee \otimes G) = 0$. Let E be a 1-dimensional sheaf on a fiber of π such that $\chi(G, E) = 0$. Then $\mathbf{R}\pi_*(G^\vee \otimes E) = 0$ if and only if E is a G-twisted semi-stable sheaf with respect to an ample divisor L on X.*

Proof. By the proof of Lemma 1.1.5 (1), we can take a decomposition

$$0 \to E_1 \to E \to E_2 \to 0 \tag{1.41}$$

such that $\mathbf{R}\pi_*(G^\vee \otimes E_1) = \pi_*(G^\vee \otimes E)$ and $\mathbf{R}\pi_*(G^\vee \otimes E_2) = R^1\pi_*(G^\vee \otimes E)[-1]$. Then $\chi(G, E_1) \ge 0 \ge \chi(G, E_2)$. Hence if E is G-twisted semi-stable, then $\pi_*(G^\vee \otimes E_1) = \pi_*(G^\vee \otimes E) = 0$, which also implies that $R^1\pi_*(G^\vee \otimes E) = 0$. Conversely if $\pi_*(G^\vee \otimes E) = R^1\pi_*(G^\vee \otimes E) = 0$, then $\pi_*(G^\vee \otimes E') = 0$ for any subsheaf E' of E. Hence E is G-twisted semi-stable. □

Corollary 1.1.26. *Assume that $\pi : X \to Y$ is the minimal resolution of a rational double point. Let H be the pull-back of an ample divisor on Y. Then a locally free sheaf G on X is a tilting generator of the category $\mathcal{C}_G$ in Lemma 1.1.5 if and only if*

(i) *$R^1\pi_*(G^\vee \otimes G) = 0$ and*
(ii) *there is no G-twisted stable sheaf E such that $\operatorname{rk} E = 0$, $\chi(G^\vee \otimes E) = 0$, $(c_1(E), H) = 0$ and $(c_1(E)^2) = -2$.*

Moreover (ii) is equivalent to $\operatorname{rk} G \not| (c_1(G), D)$ for D with $(D, H) = 0$ and $(D^2) = -2$.

Proof. Let E be a 1-dimensional G-twisted stable sheaf on X. Then E is a sheaf on the exceptional locus if and only if $(c_1(E), H) = 0$. Under this assumption, we have $\chi(E, E) = -(c_1(E)^2) > 0$. Hence $(c_1(E)^2) = -2$. By Lemma 1.1.25, we get the first part of our claim. Since $\chi(G, E) = -(c_1(G), c_1(E)) + \operatorname{rk} G\chi(E)$, we also get the second claim by [Y6, Prop. 4.6]. □

1.2. **Examples of perverse coherent sheaves.** Let $\pi : X \to Y$ be a birational map in subsection 1.1 with Assumption 1.1.4. Let G be a locally free sheaf on X such that $R^1\pi_*(G^\vee \otimes G) = 0$. We set $\mathcal{A} := \pi_*(G^\vee \otimes G)$ as before. Let F be a coherent $\mathcal{A}$-module on Y. Then $\mathbf{R}\pi_*((\pi^{-1}(F) \overset{\mathbf{L}}{\otimes}_{\pi^{-1}(\mathcal{A})} G) \otimes G^\vee) \cong F$ as an $\mathcal{A}$-module. By using the spectral sequence, we see that

$$R^p\pi_*(G^\vee \otimes H^q(\pi^{-1}(F) \overset{\mathbf{L}}{\otimes}_{\pi^{-1}(\mathcal{A})} G)) = 0,\ p+q \neq 0 \tag{1.42}$$

and we have an exact sequence

$$0 \to R^1\pi_*(G^\vee \otimes H^{-1}(\pi^{-1}(F) \overset{\mathbf{L}}{\otimes}_{\pi^{-1}(\mathcal{A})} G)) \to F \overset{\lambda}{\to} \pi_*(G^\vee \otimes H^0(\pi^{-1}(F) \overset{\mathbf{L}}{\otimes}_{\pi^{-1}(\mathcal{A})} G)) \to 0. \tag{1.43}$$

We set

$$\pi^{-1}(F) \otimes_{\pi^{-1}(\mathcal{A})} G := H^0(\pi^{-1}(E) \overset{\mathbf{L}}{\otimes}_{\pi^{-1}(\mathcal{A})} G) \in \operatorname{Coh}(X). \tag{1.44}$$

We set

$$\begin{aligned} S_0 :=& \{E \in \operatorname{Coh}(X) | \mathbf{R}\pi_*(G^\vee \otimes E) = 0\}, \\ S :=& \{E \in \operatorname{Coh}(X) | \pi_*(G^\vee \otimes E) = 0\}, \\ T :=& \{E \in \operatorname{Coh}(X) | R^1\pi_*(G^\vee \otimes E) = 0,\ \operatorname{Hom}(E, c) = 0, c \in S_0\}. \end{aligned} \tag{1.45}$$

Lemma 1.2.1. *For $E \in \operatorname{Coh}(X)$, let $\phi : \pi^{-1}(\pi_*(G^\vee \otimes E)) \otimes_{\pi^{-1}(\mathcal{A})} G \to E$ be the evaluation map.*

(1) $\mathbf{R}\pi_*(G^\vee \otimes \ker \phi) = 0$, $\pi_*(G^\vee \otimes \operatorname{coker} \phi) = 0$ *and* $R^1\pi_*(G^\vee \otimes E) \cong R^1\pi_*(G^\vee \otimes \operatorname{coker} \phi)$.

(2) (S, T) *is a torsion pair of* $\operatorname{Coh}(X)$ *and the decomposition of E is given by*

$$0 \to \operatorname{im} \phi \to E \to \operatorname{coker} \phi \to 0, \tag{1.46}$$

$\operatorname{im} \phi \in T$, $\operatorname{coker} \phi \in S$.

Proof. (1) We have a homomorphism

$$\pi_*(G^\vee \otimes E) \xrightarrow{\lambda} \pi_*(G^\vee \otimes \pi^{-1}(\pi_*(G^\vee \otimes E)) \otimes_{\pi^{-1}(\mathcal{A})} G) \xrightarrow{\pi_*(1_{G^\vee} \otimes \phi)} \pi_*(G^\vee \otimes E) \tag{1.47}$$

which is the identity. Then λ and $\pi_*(1_{G^\vee} \otimes \phi)$ are isomorphic. Hence we get $\operatorname{im} \pi_*(1_{G^\vee} \otimes \phi) = \pi_*(G^\vee \otimes \operatorname{im} \phi) = \pi_*(G^\vee \otimes E)$. Since $R^1\pi_*(G^\vee \otimes \pi^{-1}(\pi_*(G^\vee \otimes E)) \otimes_{\pi^{-1}(\mathcal{A})} G) = 0$, we get $\mathbf{R}\pi_*(G^\vee \otimes \ker \phi) = 0$. Since $R^1\pi_*(G^\vee \otimes \operatorname{im} \phi) = 0$, we also get the remaining claims.

(2) We shall prove that $\operatorname{im} \phi \in T$. If $\operatorname{im} \phi \notin T$, then there is a homomorphism $\psi : \operatorname{im} \phi \to F$ such that $F \in S$. Replacing F by $\operatorname{im} \psi$, we may assume that ψ is surjective. Since $\psi \circ \phi$ is surjective, $\operatorname{Hom}(G, F) \neq 0$, which is a contradiction. Therefore $\operatorname{im} \phi \in T$. Obviously we have $S \cap T = \{0\}$. Therefore (S, T) is a torsion pair. □

Let $\mathcal{C}(G)$ be the tilting of $\operatorname{Coh}(X)$. Then $\mathcal{C}(G)$ is the category of perverse coherent sheaves in the sense of Definition 1.1.1. Indeed we have the following.

Lemma 1.2.2. *(cf.* [VB, Prop. 3.2.5]*) Let G be a locally free sheaf on X such that $R^1\pi_*(G^\vee \otimes G) = 0$. Let $\mathcal{C}(G)$ be the associated category. Then there is a local projective generator of $\mathcal{C}(G)$.*

Proof. Let L be a line bundle on X such that $G^\vee \otimes L$ is generated by global sections and $\det(G^\vee \otimes L)$ is ample. We take a locally free resolution $0 \to L_{-1} \to L_0 \to L \to 0$ such that $R^1\pi_*(L_0^\vee \otimes G) = 0$. Then

$$\mathbf{R}\pi_*(L^\vee \otimes G)[1] = \operatorname{Cone}(\pi_*(L_0^\vee \otimes G) \to \pi_*(L_{-1}^\vee \otimes G)). \tag{1.48}$$

We take a surjective homomorphism $V \to \pi_*(L_{-1}^\vee \otimes G)$ from a locally free sheaf V on Y. Then we have a morphism $\pi^*(V) \otimes L \to \mathbf{L}\pi^*(\mathbf{R}\pi_*(L^\vee \otimes G))[1] \otimes L \to G[1]$, which induces a surjective homomorphism $V \to R^1\pi_*(L^\vee \otimes G)$. Hence we have a morphism

$$L \to G[1] \otimes \pi^*(V)^\vee \tag{1.49}$$

such that the induced homomorphism

$$V \to \pi_*(\mathcal{H}om(G[1], G[1])) \otimes V \to R^1\pi_*(L^\vee \otimes G) \tag{1.50}$$

is surjective. We set $E := \operatorname{Cone}(L \to G[1] \otimes \pi^*(V)^\vee)[-1]$. Then E is a locally free sheaf on X and $\phi : \pi^*(\pi_*(G^\vee \otimes E)) \otimes G \to E$ is surjective by our choice of L. By (1.50) and our assumption, we have $R^1\pi_*(E^\vee \otimes G) = 0$. For $F \in T$, we consider the evaluation map $\varphi : \pi^*(\pi_*(G^\vee \otimes F)) \otimes G \to F$. The proof of Lemma 1.1.5 (1) implies that $\operatorname{coker} \varphi \in S_0$. By the definition of T, $\operatorname{coker} \varphi = 0$. Thus φ is surjective. Hence $R^1\pi_*(E^\vee \otimes F) = 0$ for $F \in T$.

For $F \in S$, the surjectivity of ϕ implies that $\pi_*(E^\vee \otimes F) = 0$. If $F \notin S_0$, then $R^1\pi_*(G^\vee \otimes F) \neq 0$, which implies that $R^1\pi_*(E^\vee \otimes F) \neq 0$. Assume that $F \in S_0$. Then since $\mathbf{R}\pi_*(G^\vee \otimes F) = 0$ for $F \in S_0$, we have $R^1\pi_*(E^\vee \otimes F) \cong R^1\pi_*(L^\vee \otimes F)$. Assume that $R^1\pi_*(L^\vee \otimes F) = 0$ and $F \neq 0$. Let W be an irreducible component of $\operatorname{Supp}(F)$. Then F contains a subsheaf F' whose support is contained in W. If $W \to Y$ is generically finite, then $\pi_*(G^\vee \otimes F') \neq 0$, which is a contradiction. Therefore $\dim F' = \dim \pi(F') + 1$. For a point $y \in \pi_*(F')$, we can take a homomorphism $\psi : \mathcal{O}_X^{\oplus(\operatorname{rk} G)-1} \to G^\vee \otimes L$ such that $\psi_{|\pi^{-1}(y)}$ is injective for any point of $\pi^{-1}(y)$. Then $\operatorname{coker} \psi$ is a line bundle in a neighborhood of $\pi^{-1}(y)$. Since π is proper, there is an open neighborhood U of y such that $\operatorname{coker} \psi_{\pi^{-1}(U)}$ is a line bundle. Hence we have an exact sequence on $\pi^{-1}(U)$:

$$0 \to \mathcal{O}_{\pi^{-1}(U)}^{\oplus(\operatorname{rk} G)} \to (G^\vee \otimes L)_{|\pi^{-1}(U)} \to C \to 0, \tag{1.51}$$

where $C := \operatorname{coker} \psi_{\pi^{-1}(U)}/\mathcal{O}_{\pi^{-1}(U)}$. We may assume that $\operatorname{Supp}(C)_{|\pi^{-1}(y)}$ is a finite set. Then $\operatorname{Supp}(F' \otimes C) \to Y$ is generically finite. Hence $\pi_*(F' \otimes C \otimes L^\vee) \neq 0$, which implies that $\pi_*(F \otimes C \otimes L^\vee) \neq 0$. On the other hand, our assumptions impliy that $\mathbf{R}\pi_*(F \overset{\mathbf{L}}{\otimes} C \otimes L^\vee) = 0$. Since the spectral spectral sequence

$$E_2^{pq} = R^p\pi_*(H^q(F \overset{\mathbf{L}}{\otimes} C \otimes L^\vee)) \Rightarrow E_\infty^{p+q} = H^{p+q}(\mathbf{R}\pi_*(F \overset{\mathbf{L}}{\otimes} C \otimes L^\vee)) \tag{1.52}$$

degenerates, we have $\pi_*(F \otimes C \otimes L^\vee) = 0$, which is a contradiction. Hence $R^1\pi_*(L^\vee \otimes F) \neq 0$ for all non-zero $F \in S_0$. Then $G_1 := G \oplus E$ satisfies

$$\begin{aligned} &\pi_*(G_1^\vee \otimes F) \neq 0,\ R^1\pi_*(G_1^\vee \otimes F) = 0,\ 0 \neq F \in T \\ &\pi_*(G_1^\vee \otimes F) = 0,\ R^1\pi_*(G_1^\vee \otimes F) \neq 0,\ 0 \neq F \in S. \end{aligned} \tag{1.53}$$

Therefore G_1 is a local projective generator of $\mathcal{C}(G)$. $\square$

We set

$$S^* := \{E \in \operatorname{Coh}(X) | \pi_*(G^\vee \otimes E) = 0,\ \operatorname{Hom}(c, E) = 0, c \in S_0\},$$
$$T^* := \{E \in \operatorname{Coh}(X) | R^1\pi_*(G^\vee \otimes E) = 0\}. \tag{1.54}$$

Lemma 1.2.3. *(S^*, T^*) is a torsion pair of* $\operatorname{Coh}(X)$ *and the tilted category $\mathcal{C}(G)^*$ has a local projective generator.*

Proof. We set

$$S_0' := \{E \in \operatorname{Coh}(X) | \mathbf{R}\pi_*(G \otimes E) = 0\},$$
$$S_1 := \{E \in \operatorname{Coh}(X) | \pi_*(G \otimes E) = 0\},$$
$$T_1 := \{E \in \operatorname{Coh}(X) | R^1\pi_*(G \otimes E) = 0,\ \operatorname{Hom}(E, c) = 0, c \in S_0'\}. \tag{1.55}$$

Then (S_1, T_1) is a torsion pair of $\operatorname{Coh}(X)$ and Lemma 1.2.2 implies that the tilted category $\mathcal{C}(G^\vee)$ has a local projective generator $G^\vee \oplus E_1$, where E_1 is a locally free sheaf on X such that $\phi : \pi^*(\pi_*(G \otimes E_1)) \otimes G^\vee \to E_1$ is surjective and $R^1\pi_*(G^\vee \otimes E_1^\vee) = 0$. By Lemma 1.1.8, (S_1^D, T_1^D) is a torsion pair of $\operatorname{Coh}(X)$. We prove that $\mathcal{C}(G)^* = \mathcal{C}(G^\vee)^D$ by showing $(S_1^D, T_1^D) = (S^*, T^*)$. By the surjectivity of ϕ, we have

$$T_1^D = \{E \in \operatorname{Coh}(X) | R^1\pi_*(G^\vee \otimes E) = R^1\pi_*(E_1 \otimes E) = 0\} = T^*. \tag{1.56}$$

For a coherent sheaf E with $\pi_*(G^\vee \otimes E) = 0$, we consider $\psi : \pi^*(\pi_*(E_1 \otimes E)) \otimes E_1^\vee \to E$. Then $\operatorname{im}\psi \in T_1^D = T^*$ and $\operatorname{coker}\psi \in S_1^D$. Since $\pi_*(G^\vee \otimes \operatorname{im}\psi) = 0$, $\operatorname{im}\psi \in S_0$. Therefore if $E \in S^*$, then $\operatorname{im}\psi = 0$, which means that $E \in S_1^D$. Conversely if $E \in S_1^D$, then $S_0 \subset T_1^D$ implies that $E \in S^*$. Therefore $(S_1^D, T_1^D) = (S^*, T^*)$. $\square$

Let E_{yj}, $y \in Y_\pi$ be the irreducible objects of $\mathcal{C}$ in Lemma 1.1.13 (3).

Lemma 1.2.4. *We set $S_{0y} := \{E \in S_0 | \pi(\operatorname{Supp}(E)) = \{y\}\}$. Then $S_{0y}[1]$ is generated by $\{E_{yj} | E_{yj} \in S_0[1]\}$.*

Proof. For an exact sequence

$$0 \to E_1 \to E \to E_2 \to 0 \tag{1.57}$$

in $\mathcal{C}$, we have an exact sequence

$$0 \to \mathbf{R}\pi_*(G^\vee \otimes E_1) \to \mathbf{R}\pi_*(G^\vee \otimes E) \to \mathbf{R}\pi_*(G^\vee \otimes E_2) \to 0 \tag{1.58}$$

in $\operatorname{Coh}(Y)$. If $E \in S_0[1]$, then $\mathbf{R}\pi_*(G^\vee \otimes E_1) = \mathbf{R}\pi_*(G^\vee \otimes E_2) = 0$. Then $\mathbf{R}\pi_*(G^\vee \otimes H^{-1}(E_1)) = \mathbf{R}\pi_*(G^\vee \otimes H^{-1}(E_2)) = 0$ and $\mathbf{R}\pi_*(G^\vee \otimes H^0(E_1)) = \mathbf{R}\pi_*(G^\vee \otimes H^0(E_2)) = 0$. By the definition of T, $H^0(E_1) = H^0(E_2) = 0$. Hence $E_1, E_2 \in S_0[1]$. Therefore the claim holds. $\square$

By the construction of $\mathcal{C}(G)$ and $\mathcal{C}(G)^*$, we have the following.

Proposition 1.2.5. *We set $\mathcal{A}_0 := \pi_*(G^\vee \otimes G)$. Then we have morphisms*

$$\begin{array}{ccc} \mathcal{C}(G) & \to & \operatorname{Coh}_{\mathcal{A}_0}(Y) \\ E & \mapsto & \mathbf{R}\pi_*(G^\vee \otimes E) \end{array} \tag{1.59}$$

and

$$\begin{array}{ccc} \mathcal{C}(G)^* & \to & \operatorname{Coh}_{\mathcal{A}_0}(Y) \\ E & \mapsto & \mathbf{R}\pi_*(G^\vee \otimes E). \end{array} \tag{1.60}$$

Let $\tau^{\geq -1} : \mathbf{D}(X) \to \mathbf{D}(X)$ be the trancation morphism such that $H^p(\tau^{\geq -1}(E)) = 0$ for $p < -1$ and $H^p(\tau^{\geq -1}(E)) = H^p(E)$ for $p \geq -1$. By (1.42), we have

$$H^q(\pi^{-1}(F) \overset{\mathbf{L}}{\otimes}_{\pi^{-1}(\mathcal{A})} G) \in S_0,\ q \neq -1, 0,$$
$$\Sigma(F) := \tau^{\geq -1}(\pi^{-1}(F) \overset{\mathbf{L}}{\otimes}_{\pi^{-1}(\mathcal{A})} G) \in \mathcal{C}(G). \tag{1.61}$$

Thus we have a morphism $\Sigma : \operatorname{Coh}_{\mathcal{A}_0}(Y) \to \mathcal{C}(G)$ such that $\mathbf{R}\pi_*(G^\vee \otimes \Sigma(F)) = F$ for $F \in \operatorname{Coh}_{\mathcal{A}_0}(Y)$.

1.2.1. ${}^{p}\operatorname{Per}(X/Y)$, $p=-1,0$ *and their generalizations.* If $S_0=\{0\}$, then G is a local projective generator of $\mathcal{C}(G)$. We give examples such that $S_0 \neq \{0\}$. For $y \in Y_\pi$, we set $Z_y := \pi^{-1}(y)$ and C_{yj}, $j=1,...,s_y$ the irreducible components of Z_y. Assume that $\mathbf{R}\pi_*(\mathcal{O}_X)=\mathcal{O}_Y$. Then S_0 for $\mathcal{O}_X$ contains $\mathcal{O}_{C_{yj}}(-1)$, $y \in Y_\pi$. In this case, $\mathcal{C}(\mathcal{O}_X)$ is nothing but the category ${}^{-1}\operatorname{Per}(X/Y)$ defined by Bridgeland. We also have $\mathcal{C}(\mathcal{O}_X)^* = \mathcal{C}(\mathcal{O}_X^\vee)^D = {}^{0}\operatorname{Per}(X/Y)$. We shall study S_0 containing line bundles on C_{yj}, $y \in Y_\pi$. For this purpose, we first prepare some properties of S_0 for $\mathcal{O}_X$.

Lemma 1.2.6. (1) *Let E be a stable 1-dimensional sheaf such that* $\operatorname{Supp}(E) \subset Z_y$ *and* $\chi(E)=1$. *Then there is a curve $D \subset Z_y$ and $E \cong \mathcal{O}_D$. Conversely if $\mathcal{O}_D$ is purely 1-dimensional, $\chi(\mathcal{O}_D)=1$ and $\pi(D)=\{y\}$, then $\mathcal{O}_D$ is stable. In particular, D is a subscheme of Z_y.*

(2) *$\mathcal{O}_{Z_y}$ is stable.*

Proof. (1) Since $\chi(E)=1$, $\pi_*(E) \neq 0$. Since $\pi_*(E)$ is 0-dimensional, we have a homomorphism $\mathbb{C}_y \to \pi_*(E)$. Then we have a homomorphism $\phi: \mathcal{O}_{Z_y} = \pi^*(\mathbb{C}_y) \to E$. We denote the image by $\mathcal{O}_D$. Since $R^1\pi_*(\mathcal{O}_X)=0$, we have $H^1(X,\mathcal{O}_D)=0$. Hence $\chi(\mathcal{O}_D) \geq 1$. Since E is stable, ϕ must be surjective.

Conversely we assume that $\mathcal{O}_D$ satisfies $\chi(\mathcal{O}_D)=1$. For a quotient $\mathcal{O}_D \to \mathcal{O}_C$, $H^1(X,\mathcal{O}_C)=0$ implies that $\chi(\mathcal{O}_C) \geq 1$, which implies that $\mathcal{O}_D$ is stable.

(2) By $\mathcal{O}_{Z_y} = \pi^*(\mathbb{C}_y)$ and the surjectivity of $\mathbb{C}_y \to \pi_*(\pi^*(\mathbb{C}_y))$, we get $\chi(\mathcal{O}_{Z_y})=1$. Hence $\mathcal{O}_{Z_y}$ is stable. □

Lemma 1.2.7. (1) *Let E be a stable purely 1-dimension sheaf such that* $\pi(\operatorname{Supp}(E))=\{y\}$ *and* $\chi(E)=0$. *Then $E \cong \mathcal{O}_{C_{yj}}(-1)$.*

(2) *Let E be a 1-dimensional sheaf such that $\mathbf{R}\pi_*(E)=0$. Then E is a semi-stable 1-dimensional sheaf with $\chi(E)=0$. In particular, E is a successive extension of $\mathcal{O}_{C_{yj}}(-1)$, $y \in Y$, $1 \leq j \leq s_y$.*

Proof. (1) We set $n := \dim X$. We take a point $x \in \operatorname{Supp}(E)$. Then $\mathcal{E}xt^1_{\mathcal{O}_X}(\mathbb{C}_x, E) = \mathbb{C}_x \overset{\mathbf{L}}{\otimes} E[-n+1]$. Since E is purely 1-dimensional, $\operatorname{depth}_{\mathcal{O}_{X,x}} E_x = 1$. Hence the projective dimension of E at x is $n-1$. Then $\mathcal{T}or^{\mathcal{O}_X}_{n-1}(\mathbb{C}_x, E) = H^0(\mathbb{C}_x \overset{\mathbf{L}}{\otimes} E[-n+1]) \neq 0$. Since $\operatorname{Ext}^1(\mathbb{C}_x, E) = H^0(X, \mathcal{E}xt^1_{\mathcal{O}_X}(\mathbb{C}_x, E)) \neq 0$, we can take a non-trivial extension

$$0 \to E \to F \to \mathbb{C}_x \to 0. \tag{1.62}$$

If F is not semi-stable, then since $\chi(F)=1$, there is a quotient $F \to F'$ of F such that F' is a stable sheaf with $\chi(F') \leq 0$. Then $E \to F'$ is an isomorphism, which is a contradiction. By Lemma 1.2.6, $F = \mathcal{O}_D$. We take an integral curve $C \subset D$ containing x. Since $\mathcal{O}_D \to \mathbb{C}_x$ factor through $\mathcal{O}_C$, we have a surjective homomorphism $E \to \mathcal{O}_C(-1)$. By the stability of E, $E \cong \mathcal{O}_C(-1)$.

(2) Let F be a subsheaf of E. Then we have $\pi_*(F)=0$, which implies that $\chi(F) \leq 0$. Therefore E is semi-stable. □

We shall slightly generalize ${}^{-1}\operatorname{Per}(X/Y)$. Let G be a locally free sheaf on X.

Assumption 1.2.8. Assume that $R^1\pi_*(G^\vee \otimes G)=0$ and there are line bundles $\mathcal{O}_{C_{yj}}(b_{yj})$ on C_{yj} such that $\mathbf{R}\pi_*(G^\vee \otimes \mathcal{O}_{C_{yj}}(b_{yj}))=0$.

Lemma 1.2.9. (1) *Let E be a locally free sheaf of rank r on X such that $E_{|C_{yj}} \cong \mathcal{O}_{C_{yj}}^{\oplus r}$. Then E is the pull-back of a locally free sheaf on Y.*

(2) *$G^\vee \otimes G \cong \pi^*(\pi_*(G^\vee \otimes G))$.*

Proof. (1) We consider the map $\phi: H^0(E_{|Z_y}) \otimes \mathcal{O}_{Z_y} \to E_{|Z_y}$. For any point $x \in Z_y$, we have an exact sequence

$$0 \to F_x \to \mathcal{O}_{Z_y} \to \mathbb{C}_x \to 0 \tag{1.63}$$

such that $\mathbf{R}\pi_*(F_x)=0$. By Lemma 1.2.7 (2) and our assumption, we have $\mathbf{R}\pi_*(E \otimes F_x)=0$. Hence $H^0(E_{|Z_y}) \to H^0(E_{|\{x\}})$ is isomorphic and $H^1(E_{|Z_y})=0$. Therefore ϕ is a surjective homomorphism of locally free sheaves of the same rank, which implies that ϕ is an isomorphism. By $R^1\pi_*(E)=0$ (Lem. 1.1.16 (3)) and the surjectivity of $\pi^*(\pi_*(I_{Z_y})) \to I_{Z_y}$, $R^1\pi_*(E \otimes I_{Z_y})=0$. Hence $\pi_*(E) \to \pi_*(E_{|Z_y})$ is surjective. Then we can take a homomorphism $\mathcal{O}_U^{\oplus r} \to \pi_*(E)_{|U}$ in a neighborhood of y such that $\mathcal{O}_U^{\oplus r} \to \pi_*(E_{|Z_y})$ is surjective. Then we have a homomorphism $\pi^*(\mathcal{O}_U^{\oplus r}) \to E_{|\pi^{-1}(U)}$ which is surjective on Z_y. Since π is proper, replacing U by a small neighborhood of y, we have an isomorphism $\pi^*(\mathcal{O}_U^{\oplus r}) \to E_{|\pi^{-1}(U)}$. Therefore E is the pull-back of a locally free sheaf on Y.

(2) Since $G^\vee \otimes \mathcal{O}_{C_{yj}}(b_{yj})$ is a locally free sheaf on C_{yj} with $\mathbf{R}\pi_*(G^\vee \otimes \mathcal{O}_{C_{yj}}(b_{yj}))=0$, we have $G^\vee \otimes \mathcal{O}_{C_{yj}}(b_{yj}) \cong \mathcal{O}_{C_{yj}}(-1)^{\oplus \operatorname{rk} G}$. Hence $G_{|C_{yj}} \cong \mathcal{O}_{C_{yj}}(1)^{\oplus \operatorname{rk} G} \otimes \mathcal{O}_{C_{yj}}(b_{yj})$. Hence $G^\vee \otimes G_{|C_{yj}} \cong \mathcal{O}_{C_{yj}}^{\oplus (\operatorname{rk} G)^2}$. By (1), we get the claim.

□

Lemma 1.2.10. *For $E \in \mathrm{Coh}(X)$, we have*

$$\pi^{-1}(\pi_*(G^\vee \otimes E)) \otimes_{\pi^{-1}(\mathcal{A})} G \otimes_{\mathcal{O}_X} G^\vee \cong \pi^*\pi_*(G^\vee \otimes E). \tag{1.64}$$

Proof. By Lemma 1.2.9, we get

$$\begin{aligned}\pi^{-1}(\pi_*(G^\vee \otimes E)) \otimes_{\pi^{-1}(\mathcal{A})} G \otimes_{\mathcal{O}_X} G^\vee \cong & \pi^{-1}(\pi_*(G^\vee \otimes E)) \otimes_{\pi^{-1}(\mathcal{A})} \pi^{-1}(\pi_*(G \otimes_{\mathcal{O}_X} G^\vee)) \otimes_{\pi^{-1}(\mathcal{O}_Y)} \mathcal{O}_X \\ \cong & \pi^{-1}(\pi_*(G^\vee \otimes E)) \otimes_{\pi^{-1}(\mathcal{O}_Y)} \mathcal{O}_X \\ = & \pi^*(\pi_*(G^\vee \otimes E)).\end{aligned} \tag{1.65}$$

Therefore the claims hold. ☐

Lemma 1.2.11. *$\mathcal{A}$-module $\pi_*(G^\vee \otimes \mathbb{C}_x)$ does not depend on the choice of $x \in \pi^{-1}(y)$. We set*

$$A_y := \pi^{-1}(\pi_*(G^\vee \otimes \mathbb{C}_x)) \otimes_{\pi^{-1}(\mathcal{A})} G, \ x \in Z_y. \tag{1.66}$$

Proof. For the exact sequence

$$0 \to \mathcal{O}_{C_{yj}}(b_{yj}) \to \mathcal{O}_{C_{yj}}(b_{yj}+1) \to \mathbb{C}_x \to 0, \tag{1.67}$$

we have $\pi_*(G^\vee \otimes \mathcal{O}_{C_{yj}}(b_{yj}+1)) \cong \pi_*(G^\vee \otimes \mathbb{C}_x)$. Hence $\pi_*(G^\vee \otimes \mathbb{C}_x)$ does not depens on the choice of $x \in Z_y$. ☐

Lemma 1.2.12. (1) *A_y is a unique line bundle on Z_y such that $A_{y|C_{yj}} \cong \mathcal{O}_{C_{yj}}(b_{yj}+1)$.*
(2) *$G^\vee \otimes A_y \cong \mathcal{O}_{Z_y}^{\oplus \mathrm{rk}\, G}$.*

Proof. By Lemma 1.2.10, $G^\vee \otimes A_y \cong \pi^*(\pi_*(G^\vee \otimes \mathbb{C}_x)) \cong \mathcal{O}_{Z_y}^{\oplus \mathrm{rk}\, G}$. Thus (2) holds. Since $G_{|Z_y}$ is a locally free sheaf on Z_y, A_y is a line bundle on Z_y. Then $A_y^{\otimes \mathrm{rk}\, G} \cong \det G_{|Z_y}$. Since the restriction map $\mathrm{Pic}(Z_y) \to \prod_j \mathrm{Pic}(C_{yj})$ is bijective and $\mathrm{Pic}(C_{yj}) \cong \mathbb{Z}$, $G_{|C_{yj}} \cong \mathcal{O}_{C_{yj}}(b_{yj}+1)^{\oplus \mathrm{rk}\, G}$ imply the claim (1). ☐

Lemma 1.2.13. *For a coherent sheaf E with $\mathrm{Supp}(E) \subset Z_y$, $\chi(G, E) \in \mathbb{Z}\,\mathrm{rk}\, G$.*

Proof. We note that $K(Z_y)$ is generated by $\mathcal{O}_{C_{yj}}(b_{yj})$ and $\mathbb{C}_x$. For E with $\mathrm{Supp}(E) \subset Z_y$, we have a filtration $0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = E$ such that $F_i/F_{i-1} \in \mathrm{Coh}(Z_y)$. Hence the claim follows from $\chi(G, \mathcal{O}_{C_{yj}}(b_{yj})) = 0$ and $\chi(G, \mathbb{C}_x) = \mathrm{rk}\, G$. ☐

Lemma 1.2.14. (1) *Let E be a G-twisted stable 1-dimensional sheaf such that $\mathrm{Supp}(E) \subset Z_y$ and $\chi(G, E) = \mathrm{rk}\, G$. Then there is a subscheme C of Z_y such that $\chi(\mathcal{O}_C) = 1$ and $E \cong A_y \otimes \mathcal{O}_C$. Conversely for a subscheme C of Z_y such that $\mathcal{O}_C$ is 1-dimensional, $\chi(\mathcal{O}_C) = 1$, $E = A_y \otimes \mathcal{O}_C$ is a G-twisted stable sheaf with $\chi(G, E) = \mathrm{rk}\, G$ and $\pi(\mathrm{Supp}(E)) = \{y\}$.*
(2) *A_y is G-twisted stable.*

Proof. (1) We choose an exact sequence

$$0 \to K \to E \to \mathbb{C}_x \to 0. \tag{1.68}$$

Since E is a G-twisted stable 1-dimensional sheaf with $\chi(G, E) = \mathrm{rk}\, G$, K is a G-twisted semi-stable sheaf with $\chi(G, K) = 0$. If $\pi_*(G^\vee \otimes K) \neq 0$, then we have a non-zero homomorphism $\phi : \pi^{-1}(\pi_*(G^\vee \otimes K)) \otimes_{\pi^{-1}(\mathcal{A})} G \to K$ such that $\pi_*(G^\vee \otimes \mathrm{im}\, \phi) = \pi_*(G^\vee \otimes K)$. Since $R^1\pi_*(G^\vee \otimes \mathrm{im}\, \phi) = 0$, $\chi(G, \mathrm{im}\, \phi) > 0$, which is a contradiction. Therefore $\pi_*(G^\vee \otimes K) = 0$. Hence $\xi : \pi_*(G^\vee \otimes E) \to \pi_*(G^\vee \otimes \mathbb{C}_x)$ is injective. Since $\dim H^0(Y, \pi_*(G^\vee \otimes E)) \geq \chi(G, E) = \mathrm{rk}\, G$, ξ is an isomorphism. Then we have a homomorphism $\psi : A_y \to E$. Since $\pi_*(G^\vee \otimes \mathrm{im}\, \psi) = \pi_*(G^\vee \otimes E)$ and $R^1\pi_*(G^\vee \otimes \mathrm{im}\, \psi) = 0$, we get $\mathrm{im}\, \psi = E$. Since $E \otimes A_y^D$, $A_y^D := \mathcal{H}om(A_y, \mathcal{O}_{Z_y})$ is a quotient of $\mathcal{O}_{Z_y}$, there is a subscheme C of Z_y such that $E \otimes A_y^D \cong \mathcal{O}_C$. Since $\chi(G, E) = \chi(G, A_y \otimes \mathcal{O}_C) = \chi(\mathcal{O}_C^{\oplus \mathrm{rk}\, G})$, we have $\chi(\mathcal{O}_C) = 1$.

Conversely for $E \otimes A_y^\vee \cong \mathcal{O}_C$ such that $\mathcal{O}_C$ is 1-dimensional, $C \subset Z_y$ and $\chi(\mathcal{O}_C) = 1$, we consider a quotient $E \to F$. Then $F = A_y \otimes \mathcal{O}_D$, $D \subset C$. Since $R^1\pi_*(G^\vee \otimes F) = 0$ and $G^\vee \otimes A_y \otimes \mathcal{O}_D \cong \mathcal{O}_D^{\oplus \mathrm{rk}\, G}$, we get $\chi(G, F) \geq \mathrm{rk}\, G$. From this fact, we first see that E is purely 1-dimensional, and then we see that G-twisted stable.

(2) follows from (1) and $\chi(\mathcal{O}_{Z_y}) = 1$. ☐

Lemma 1.2.15. *Let E be a G-twisted stable purely 1-dimension sheaf such that $\pi(\mathrm{Supp}(E)) = \{y\}$ and $\chi(G, E) = 0$. Then $E \cong A_y \otimes \mathcal{O}_{C_{yj}}(-1) \cong \mathcal{O}_{C_{yj}}(b_{yj})$.*

Proof. We set $n := \dim X$. We take a point $x \in \mathrm{Supp}(E)$. Then $\mathcal{E}xt^1_{\mathcal{O}_X}(\mathbb{C}_x, E) = \mathbb{C}_x \overset{\mathbf{L}}{\otimes} E[-n+1]$. Since E is purely 1-dimensional, $\mathrm{depth}_{\mathcal{O}_{X,x}} E_x = 1$. Hence the projective dimension of E at x is $n-1$. Then

$\mathcal{T}or_{n-1}^{\mathcal{O}_X}(\mathbb{C}_x, E) = H^0(\mathbb{C}_x \overset{\mathbf{L}}{\otimes} E[-n+1]) \neq 0$. Since $\operatorname{Ext}^1(\mathbb{C}_x, E) = H^0(X, \mathcal{E}xt^1_{\mathcal{O}_X}(\mathbb{C}_x, E)) \neq 0$, we can take a non-trivial extension

$$0 \to E \to F \to \mathbb{C}_x \to 0. \tag{1.69}$$

If F is not G-twisted semi-stable, then since $\chi(G, F) = \operatorname{rk} G$, there is a quotient $F \to F'$ of F such that F' is a G-twisted stable sheaf with $\chi(G, F') \leq 0$. Then $E \to F'$ is an isomorphism, which is a contradiction. By Lemma 1.2.14, F is a quotient of A_y. Thus we may write $F = A_y \otimes \mathcal{O}_D$, where D is a subscheme of Z_y. We take an integral curve $C \subset D$ containing x. Since $\mathcal{O}_D \to \mathbb{C}_x$ factor through $\mathcal{O}_C$, we have a surjective homomorphism $E \to A_y \otimes \mathcal{O}_C(-1)$. By the stability of E, $E \cong A_y \otimes \mathcal{O}_C(-1)$. □

Lemma 1.2.16. *Let E be a 1-dimensional sheaf such that $\chi(G, E) = 0$ and $\pi(\operatorname{Supp}(E)) = \{y\}$. Then the following conditions are equivalent.*

(1) $\mathbf{R}\pi_*(G^\vee \otimes E) = 0$.
(2) *E is a G-twisted semi-stable 1-dimensional sheaf with $\pi(\operatorname{Supp}(E)) = \{y\}$.*
(3) *E is a successive extension of $A_y \otimes \mathcal{O}_{C_{yj}}(-1)$, $1 \leq j \leq s_y$.*

Proof. Lemma 1.1.25 gives the equivalence of (1) and (2). The equivalence of (2) and (3) follows from Lemma 1.2.15. □

Lemma 1.2.17. *Let E be a 1-dimensional sheaf such that $\pi_*(G, E) = 0$. Then there is a homomorphism $E \to A_y \otimes \mathcal{O}_{C_{yj}}(-1)$. In particular, E is generated by subsheaves of $A_y \otimes \mathcal{O}_{C_{yj}}(-1)$, $y \in Y$, $1 \leq j \leq s_y$.*

Proof. Since $\pi(\operatorname{Supp}(E))$ is 0-dimensional, we have a decomposition $E = \oplus_i E_i$, $\operatorname{Supp}(E_i) \cap \operatorname{Supp}(E_j) = \emptyset$, $i \neq j$. So we may assume that $\pi(\operatorname{Supp}(E))$ is a point. We note that $\chi(G, E) \leq 0$. If $\chi(G, E) = 0$, then $\chi(R^1\pi_*(G^\vee \otimes E)) = 0$. Since $\dim E = 1$ and $\pi_*(G^\vee \otimes E) = 0$, we get $\dim \pi(\operatorname{Supp}(E)) = 0$. Then we have $R^1\pi_*(G^\vee \otimes E) = 0$. Hence the claim follows from Lemma 1.2.16. We assume that $\chi(G, E) < 0$. Let

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E \tag{1.70}$$

be a filtration such that $E_i := F_i/F_{i-1}$, $1 \leq i \leq s$ are G-twisted stable and $\chi(G, E_i)/(c_1(E_i), L) \leq \chi(G, E_{i-1})/(c_1(E_{i-1}), L)$, where L is an ample divisor on X. Since $\pi_*(G^\vee \otimes E) = 0$ for any G-twisted stable 1-dimensional sheaf E on a fiber with $\chi(G, E) \leq 0$, replacing E by a G-twisted stable sheaf E_s, we may assume that E is G-twisted stable. We take a non-trivial extension

$$0 \to E \to F \to \mathbb{C}_x \to 0. \tag{1.71}$$

Then F is purely 1-dimensional and $\chi(G, F) = \chi(G, E) + \operatorname{rk} G \leq 0$ by Lemma 1.2.13. Assume that there is a quotient $F \to F'$ of F such that F' is a G-twisted stable sheaf with $\chi(G, F')/(c_1(F'), L) < \chi(G, F)/(c_1(F), L) \leq 0$. Then $\phi : E \to F'$ is surjective over $X \setminus \{x\}$. Hence $\chi(G, F')/(c_1(F'), L) \geq \chi(G, \operatorname{im} \phi)/(c_1(\operatorname{im} \phi), L) \geq \chi(G, E)/(c_1(E), L)$. Since $(c_1(F'), L) \leq (c_1(F), L) = (c_1(E), L)$, we get $\chi(G, F') \geq \chi(G, E)(c_1(F'), L)/(c_1(E), L) \geq \chi(G, E)$. If $\chi(G, F') = \chi(G, E)$, then ϕ is an isomorphism. Since the extension is non-trivial, this is a contradiction. Therefore F is G-twisted semi-stable or $\chi(G, F') > \chi(G, E)$. Thus we get a homomorphism $\psi : E \to E'$ such that E' is a stable sheaf with $\chi(G, E) < \chi(G, E') < 0$ and ψ is surjective in codimension $n-1$. By the induction on $\chi(G, E)$, we get the claim. □

Lemma 1.2.18. *For a point $y \in Y_\pi$, let E be a 1-dimensional sheaf on X satisfying the following two conditions:*

(i) $\operatorname{Hom}(E, A_y \otimes \mathcal{O}_{C_{yj}}(-1)) = \operatorname{Ext}^1(E, A_y \otimes \mathcal{O}_{C_{yj}}(-1)) = 0$ *for all j.*
(ii) *There is an exact sequence*

$$0 \to F \to E \to \mathbb{C}_x \to 0 \tag{1.72}$$

such that F is a G-twisted semi-stable 1-dimensional sheaf with $\pi(\operatorname{Supp}(F)) = \{y\}$, $\chi(G, F) = 0$ and $x \in Z_y$.

Then $E \cong A_y$. Conversely, $E := A_y$ satisfies (i) and (ii).

Proof. We first prove that A_y satisfies (i) and (ii). For the exact sequence

$$0 \to F' \to A_y \to \mathbb{C}_x \to 0, \tag{1.73}$$

we have $\mathbf{R}\pi_*(G, F') = 0$. Hence (ii) holds by Lemma 1.2.16. (i) follows from Lemma 1.1.16. Conversely we assume that E satisfies (i) and (ii). By (ii), $\pi_*(G^\vee \otimes E) \cong \pi_*(G^\vee \otimes \mathbb{C}_x)$ and $R^1\pi_*(G^\vee \otimes E) = 0$. By (i), Lemma 1.2.1 and Lemma 1.2.16, $\pi^{-1}(\pi_*(G^\vee \otimes E)) \otimes_{\pi^{-1}(\mathcal{O}_Y)} G \to E$ is surjective. Hence we have an exact sequence

$$0 \to F' \to A_y \to E \to 0, \tag{1.74}$$

where F' is a G-twisted semi-stable 1-dimensional sheaf with $\chi(G, F') = 0$. Since $\operatorname{Ext}^1(E, A_y \otimes \mathcal{O}_{C_{yj}}(-1)) = 0$ for all j, $A_y \cong E \oplus F'$, which implies that $A_y \cong E$. □

We set

$$E_{yj} := \begin{cases} A_y, & j=0, \\ A_y \otimes \mathcal{O}_{C_{yj}}(-1)[1], & j>0. \end{cases} \tag{1.75}$$

Proposition 1.2.19. *([VB])*

(1) *E_{yj}, $j=0,...,s_y$ are irreducible objects of $\mathcal{C}(G)$.*

(2) *$\mathbb{C}_x$, $x \in \pi^{-1}(y)$ is generated by E_{yj}. In particular, irreducible objects of $\mathcal{C}(G)$ are*

$$\mathbb{C}_x, (x \in X \setminus \pi^{-1}(Y_\pi)), \quad E_{yj}, (y \in Y_\pi, j=0,1,...,s_y). \tag{1.76}$$

Proof. (1) Assume that there is an exact sequence in $\mathcal{C}(G)$:

$$0 \to E_1 \to A_y \to E_2 \to 0. \tag{1.77}$$

Since $H^{-1}(E_1)=0$, $E_1 \in T$ and $\pi_*(G^\vee \otimes E_1) \cong \pi_*(G^\vee \otimes A_y) = \mathbb{C}_y^{\oplus \operatorname{rk} G}$. Hence we have a non-zero morphism $A_y \to E_1$. Since $\operatorname{Hom}(A_y, A_y) \cong \mathbb{C}$, $E_1 \cong A_y$ and $E_2 = 0$. For $A_y \otimes \mathcal{O}_{C_{yj}}(-1)[1]$, assume that there is an exact sequence in $\mathcal{C}(G)$:

$$0 \to E_1 \to A_y \otimes \mathcal{O}_{C_{yj}}(-1)[1] \to E_2 \to 0. \tag{1.78}$$

Since $H^0(E_2)=0$, we have $E_2[-1] \in S$. Then Lemma 1.2.17 implies that we have a non-zero morphism $E_2 \to A_y \otimes \mathcal{O}_{C_{yj}}(-1)[1]$. Since $\operatorname{Hom}(A_y \otimes \mathcal{O}_{C_{yj}}(-1)[1], A_y \otimes \mathcal{O}_{C_{yj}}(-1)[1]) = \mathbb{C}$, we get $E_1 = 0$. Therefore $A_y \otimes \mathcal{O}_{C_{yj}}(-1)[1]$ is irreducible. □

We give a characterization of T.

Proposition 1.2.20. (1) *For $E \in \operatorname{Coh}(X)$, the following are equivalent.*

(a) *$E \in T$.*

(b) *$\operatorname{Hom}(E, A_y \otimes \mathcal{O}_{C_{yj}}(-1)) = 0$ for all y, j.*

(c) *$\phi : \pi^{-1}(\pi_*(G^\vee \otimes E)) \otimes_{\pi^{-1}(\mathcal{A})} G \to E$ is surjective.*

(2) *If (c) holds, then $\ker \phi \in S_0$.*

Proof. (1) is a consequence of Lemma 1.2.1 and Lemma 1.1.17.

(2) The claim follows from Lemma 1.2.1. □

We note that $G \otimes \mathcal{H}om_{\mathcal{O}_{Z_y}}(A_y, \mathcal{O}_{Z_y}) \cong \mathcal{O}_{Z_y}^{\oplus \operatorname{rk} G}$. Then we have $\mathcal{H}om_{\mathcal{O}_{Z_y}}(A_y, \mathcal{O}_{Z_y}) \cong \pi^{-1}(\pi_*(G \otimes \mathbb{C}_x)) \otimes_{\pi^{-1}(\mathcal{A})} G^\vee$. We set

$$E_{yj}^* := \begin{cases} A_y \otimes \omega_{Z_y}[1], & j=0, \\ A_y \otimes \mathcal{O}_{C_{yj}}(-1), & j>0. \end{cases} \tag{1.79}$$

Then we also have the following.

Proposition 1.2.21. [VB]

(1) *E_{yj}^*, $j=0,...,s_y$ are irreducible objects of $\mathcal{C}(G)^*$.*

(2) *$\mathbb{C}_x$, $x \in \pi^{-1}(y)$ is generated by E_{yj}^*. In particular, irreducible objects of $\mathcal{C}(G)^*$ are*

$$\mathbb{C}_x, (x \in X \setminus \pi^{-1}(Y_\pi)), \quad E_{yj}^*, (y \in Y_\pi, j=0,1,...,s_y). \tag{1.80}$$

Lemma 1.2.22. *For a point $y \in Y_\pi$, let E be a 1-dimensional sheaf on X satisfying the following two conditions:*

(i) *$\operatorname{Hom}(A_y \otimes \mathcal{O}_{C_{yj}}(-1), E) = \operatorname{Ext}^1(A_y \otimes \mathcal{O}_{C_{yj}}(-1), E) = 0$ for all j.*

(ii) *There is an exact sequence*

$$0 \to E \to F \to \mathbb{C}_x \to 0 \tag{1.81}$$

such that F is a G-twisted semi-stable 1-dimensional sheaf with $\pi(\operatorname{Supp}(F)) = \{y\}$, $\chi(G,F)=0$ and $x \in Z_y$.

Then $E \cong A_y \otimes \omega_{Z_y}$.

Proof. We set $n := \dim X$. For a purely 1-dimensional sheaf E on X, $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(E, K_X[n-1]) \in \operatorname{Coh}(X)$ and $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(E, K_X[n-1]) = \mathcal{H}om_{\mathcal{O}_C}(E, \omega_C)$ if E is a locally free sheaf on a curve without embedded primes. Hence the claim follows from Lemma 1.2.18. □

1.3. **Families of perverse coherent sheaves.** We shall explain families of complexes which correspond to families of $\mathcal{A}$-modules via Morita equivalence. Let $f : X \to S$ and $g : Y \to S$ be flat families of projective varieties parametrized by a scheme S and $\pi : X \to Y$ an S-morphism. Let $\mathcal{O}_Y(1)$ be a relatively ample line bundle over $Y \to S$. We assume that

(i) $X \to S$ is a smooth family,

(ii) there is a locally free sheaf G on X such that $G_s := G_{|f^{-1}(s)}$, $s \in S$ are local projective generators of a family of abelian categories $\mathcal{C}_s \subset \mathbf{D}(X_s)$ and

(iii) $\dim \pi^{-1}(y) \le 1$ for all $y \in Y$, i.e., π satisfies Assumption 1.1.4.

Then $\mathcal{C}_s$ is a tilting of $\mathrm{Coh}(X_s)$.

Remark 1.3.1. (i), (ii) and (iii) imply that

(iv) $R^1\pi_*(G^\vee \otimes G) = 0$.

(v)

$$\{E \in \mathrm{Coh}(X) | \mathbf{R}\pi_*(G^\vee \otimes E) = 0\} = 0. \tag{1.82}$$

Thus G defines a tilting $\mathcal{C}$ of $\mathrm{Coh}(X)$.

Indeed if $E \in \mathrm{Coh}(X)$ satisfies $\mathbf{R}\pi_*(G^\vee \otimes E) = 0$, then the projection formula implies that $\mathbf{R}\pi_*(G^\vee \otimes E \overset{\mathbf{L}}{\otimes} \mathbf{L}f^*(\mathbb{C}_s)) = \mathbf{R}\pi_*(G^\vee \otimes E) \overset{\mathbf{L}}{\otimes} \mathbf{L}g^*(\mathbb{C}_s) = 0$ for all $s \in S$. Then $\mathbf{R}\pi_*(G^\vee \otimes H^p(E \overset{\mathbf{L}}{\otimes} \mathbf{L}f^*(\mathbb{C}_s))) = 0$ for all p and $s \in S$. By (ii), $H^p(E \overset{\mathbf{L}}{\otimes} \mathbf{L}f^*(\mathbb{C}_s)) = 0$ for all p and $s \in S$. Therefore (v) holds. (iv) is obvious. Conversely if (i), (iii), (iv) and (v) hold, then (ii) holds. So we may replace (ii) by (iv) and (v).

For a morphism $T \to S$, we set $X_T := X \times_S T$, $Y_T := Y \times_S T$ and $\pi_T := \pi \times \mathrm{id}_T$.

Definition 1.3.2. (1) A family of objects in $\mathcal{C}_s, s \in S$ means a bounded complex $F^\bullet$ of coherent sheaves on X such that F^i are flat over S and $F^\bullet_s \in \mathcal{C}_s$ for all $s \in S$.

(2) A family of local projective generators is a locally free sheaf G on X such that $G_s := G_{|f^{-1}(s)}$, $s \in S$ are local projective generators of a family of abelian categories $\mathcal{C}_s$.

Remark 1.3.3. If $F^\bullet_s \in \mathrm{Coh}(X_s)$ for all $s \in S$, then $F^\bullet$ is isomorphic to a coherent sheaf on X which is flat over S.

Lemma 1.3.4. *For a family $F^\bullet$ of objects in $\mathcal{C}_s$, $s \in S$, there is a complex $\widetilde{F}^\bullet$ such that (i) $\widetilde{F}^i_s \in \mathcal{C}_s$, $s \in S$, (ii) $\widetilde{F}^i$ are flat over S, and (iii)$F^\bullet \cong \widetilde{F}^\bullet$.*

Proof. We set $d := \dim X_s, s \in S$. For the bounded complex $F^\bullet$, we take a locally free resolution of $\mathcal{O}_X$

$$0 \to V_{-d} \to \cdots \to V_{-1} \to V_0 \to \mathcal{O}_X \to 0 \tag{1.83}$$

such that $R^k\pi_*((G^\vee \otimes V_i^\vee \otimes F^j)_s) = 0$, $k > 0$ for $0 \le i \le d-1$ and all j. Since $X \to Y$ is projective, we can take such a resolution. Then $R^k\pi_*((G^\vee \otimes V_{-d}^\vee \otimes F^j)_s) = 0$, $k > 0$ for all j. Therefore we have an isomorphism $F^\bullet \cong V_\bullet^\vee \otimes F^\bullet$ such that $(V_\bullet^\vee \otimes F^\bullet)^i$ are S-flat and $(V_\bullet^\vee \otimes F^\bullet)^i_s = \oplus_{p+q=i} V_{-p}^\vee \otimes F^q_s \in \mathcal{C}_s$ for all $s \in S$. ∎

Proposition 1.3.5. (1) *Let $F^\bullet$ be a family of objects in $\mathcal{C}_s$, $s \in S$. Then we get*

$$F^\bullet \cong \mathrm{Cone}(E_1 \to E_2), \tag{1.84}$$

where $E_i \in \mathrm{Coh}(X)$ are flat over S and $(E_i)_s \in \mathcal{C}_s$, $s \in S$.

(2) *Let $F^\bullet$ be a family of objects in $\mathcal{C}_s$, $s \in S$. Then we have a complex*

$$G(-n_1) \otimes f^*(U_1) \to G(-n_2) \otimes f^*(U_2) \to F^\bullet \to 0 \tag{1.85}$$

whose restriction to $s \in S$ is exact in $\mathcal{C}_s$, where U_1, U_2 are locally free sheaves on S.

(3) *Let F be an $\mathcal{A}$-module flat over S. Then we can attach a family E of objects in $\mathcal{C}_s$, $s \in S$ such that $\mathbf{R}\pi_*(G^\vee \otimes E) = F$. The correspondence is functorial and E is unique in $\mathbf{D}(X)$. We denote E by $\pi^{-1}(F) \otimes_{\pi^{-1}(\mathcal{A})} G$.*

Proof. (1) We may assume that (i), (ii), (iii) in Lemma 1.3.4 hold for $F^\bullet$. We take a sufficiently large n with $\mathrm{Hom}_f(G(-n), F^j[i]) = 0$, $i > 0$ for all j. Then $W^j := \mathrm{Hom}_f(G(-n), F^j)$ are locally free sheaves. Let $W^\bullet := \mathbf{R}\,\mathrm{Hom}_f(G(-n), F^\bullet)$ be the complex defined by $W^j, j \in \mathbb{Z}$. Then we have a morphism $G(-n) \otimes f^*(W^\bullet) \to F^\bullet$. Since $F^\bullet_s \in \mathcal{C}_s$, $s \in S$, $\mathrm{Hom}(G_s(-n), F^\bullet_s[i]) = 0$ for $i \ne 0$ and all $s \in S$. Then the base change theorem implies that $U := \mathrm{Hom}_f(G(-n), F^\bullet)$ is a locally free sheaf on S and $\mathrm{Hom}_f(G(-n), F^\bullet)_s \cong \mathrm{Hom}(G(-n)_s, F^\bullet_s)$. Hence $G(-n) \otimes f^*(W^\bullet) \cong G(-n) \otimes f^*(U)$, which defines a family of morphisms

$$G(-n) \otimes f^*(U) \to F^\bullet. \tag{1.86}$$

Since $F_s^\bullet \in \mathcal{C}_s$ for all $s \in S$, $\mathbf{R}\pi_*(G^\vee \otimes F^\bullet)$ is a coherent sheaf on Y which is flat over S, and $g^* g_*(\pi_*(G^\vee \otimes F^\bullet)(n)) \to \pi_*(G^\vee \otimes F^\bullet)(n)$ is surjective in $\operatorname{Coh}(Y)$ for $n \gg 0$. Since $W^\bullet \cong g_*(\pi_*(G^\vee \otimes F^\bullet)(n))$, the homomorphism

$$\pi_*(G^\vee \otimes G)(-n) \otimes g^*(U) \to \pi_*(G^\vee \otimes F^\bullet) \tag{1.87}$$

in $\operatorname{Coh}(Y)$ is surjective for $n \gg 0$. Thus we have a family of exact sequences

$$0 \to E^\bullet \to G(-n) \otimes f^*(U) \to F^\bullet \to 0 \tag{1.88}$$

in $\mathcal{C}_s$, $s \in S$. Since $G \in \operatorname{Coh}(X)$, we have $E^\bullet \in \operatorname{Coh}(X)$ which is flat over S. (2) is a consequence of (1).

(3) We take a resolution of F

$$\cdots \overset{d^{-3}}{\to} g^*(U_{-2}) \otimes \mathcal{A}(-n_2) \overset{d^{-2}}{\to} g^*(U_{-1}) \otimes \mathcal{A}(-n_1) \overset{d^{-1}}{\to} g^*(U_0) \otimes \mathcal{A}(-n_0) \to F \to 0, \tag{1.89}$$

where U_i are locally free sheaves on S. Then we have a complex

$$\cdots \overset{\tilde{d}^{-3}}{\to} f^*(U_{-2}) \otimes G(-n_2) \overset{\tilde{d}^{-2}}{\to} f^*(U_{-1}) \otimes G(-n_1) \overset{\tilde{d}^{-1}}{\to} f^*(U_0) \otimes G(-n_0). \tag{1.90}$$

By the Morita equivalence (Proposition 1.1.3), we have $\operatorname{im} \tilde{d}_s^{-i} = \ker \tilde{d}_s^{-i+1}$ in $\mathcal{C}_s$ for all $s \in S$. Let $\operatorname{coker} \tilde{d}^{-2}$ be the cokernel of $\tilde{d}^{-2}$ in $\operatorname{Coh}(X)$. Then by Lemma 1.3.6 below, $\operatorname{coker} \tilde{d}^{-2}$ is flat over S, $(\operatorname{coker} \tilde{d}^{-2})_s = \operatorname{coker}(\tilde{d}_s^{-2}) \in \mathcal{C}_s$ and

$$E := \operatorname{Cone}(\operatorname{coker} \tilde{d}^{-2} \to f^*(U_0) \otimes G(-n_0)) \tag{1.91}$$

is a family of objects in $\mathcal{C}_s$. By the construction, we have $E_s = \pi^{-1}(F_s) \otimes_{\pi^{-1}(\mathcal{A}_s)} G_s$. It is easy to see the class of E in $\mathbf{D}(X)$ does not depend on the choice of the resolution (1.89) (cf. [B-S, Lem. 14]). □

Lemma 1.3.6. *Let E^i, $0 \le i \le 3$ be coherent sheaves on X which are flat over S. Let*

$$E^0 \overset{d^0}{\to} E^1 \overset{d^1}{\to} E^2 \overset{d^2}{\to} E^3 \tag{1.92}$$

be a complex in $\operatorname{Coh}(X)$.

1. *If $\ker d_s^1 = \operatorname{im} d_s^0$ in $\operatorname{Coh}(X_s)$, then $(\operatorname{im} d^1)_s \to E_s^2$ is injective. In particular if $\ker d_s^1 = \operatorname{im} d_s^0$ in $\operatorname{Coh}(X_s)$ for all $s \in S$, then $\operatorname{coker} d^1, \operatorname{im} d^1, \ker d^1$ in $\operatorname{Coh}(X)$ are flat over S and $\operatorname{im} d^0 = \ker d^1$.*
2. *Assume that $E_s^i \in \mathcal{C}_s$ for all $s \in S$. We denote the kernel, cokernel and the image of d_s^i in $\mathcal{C}_s$ by $\ker_{\mathcal{C}_s} d_s^i, \operatorname{coker}_{\mathcal{C}_s} d_s^i$ and $\operatorname{im}_{\mathcal{C}_s} d_s^i$ respectively. If $E_s^i \in \mathcal{C}_s$ and $\ker_{\mathcal{C}_s} d_s^i = \operatorname{im}_{\mathcal{C}_s} d_s^{i-1}$, $i = 1, 2$ in $\mathcal{C}_s$ for all s, then $\operatorname{im}_{\mathcal{C}_s} d_s^{i-1}$ coincide with the image of d_s^{i-1} in $\operatorname{Coh}(X_s)$ for $i = 1, 2$ and $\ker_{\mathcal{C}_s} d_s^1$ coincides with the kernel of d_s^1 in $\operatorname{Coh}(X_s)$. In particular, $\overline{E}^\bullet : E^2/d^1(E^1) \to E^3$ is a family of objects in $\mathcal{C}_s$ and we get an exact triangle:*

$$\ker d^0 \to E^\bullet \to \overline{E}^\bullet \to \ker d^0[1] \tag{1.93}$$

where $\ker d^0$ is the kernel of d^0 in $\operatorname{Coh}(X)$, which is flat over S.

Proof. (1) Let K be the kernel of $\xi : (\operatorname{im} d^1)_s \to E_s^2$. Then we have an exact sequence

$$(\ker d^1)_s \to \ker(d_s^1) \to K \to 0. \tag{1.94}$$

Since the image of $E_s^0 \to (\ker d^1)_s \to E_s^1$ is $d_s^0(E_s^0) = \ker(d_s^1)$, $K = 0$. The other claims are easily follows from this.

(2) By our assumption, $\operatorname{im}_{\mathcal{C}_s} d_s^i = \operatorname{coker}_{\mathcal{C}_s} d_s^{i-1}$ for $i = 1, 2$. Since $\operatorname{im}_{\mathcal{C}_s} d_s^i$ is a subobject of E_s^{i+1} for $i = 0, 1, 2$, $\operatorname{im}_{\mathcal{C}_s} d_s^i \in \operatorname{Coh}(X_s)$ for $i = 0, 1, 2$ and $H^{-1}(\operatorname{coker}_{\mathcal{C}_s} d_s^{i-1}) = H^{-1}(\operatorname{im}_{\mathcal{C}_s} d_s^i) = 0$ for $i = 1, 2$. Then $H^0(\operatorname{im}_{\mathcal{C}_s} d_s^{i-1}) \to H^0(E_s^i)$ is injective for $i = 1, 2$, which implies that $\operatorname{im}_{\mathcal{C}_s} d_s^{i-1}$ is the image of d_s^{i-1} in $\operatorname{Coh}(X_s)$ for $i = 1, 2$. By the exact sequence

$$0 \to H^0(\ker_{\mathcal{C}_s} d_s^1) \to H^0(E_s^1) \to H^0(\operatorname{im}_{\mathcal{C}_s} d_s^1) \to 0 \tag{1.95}$$

and the injectivity of $H^0(\operatorname{im}_{\mathcal{C}_s} d_s^1) \to H^0(E_s^2)$, $\ker_{\mathcal{C}_s} d_s^1$ is the kernel of d_s^1 in $\operatorname{Coh}(X_s)$. Then the other claims follow from (1). □

1.3.1. *Quot-schemes.*

Lemma 1.3.7. *Let $\mathcal{A}$ be an $\mathcal{O}_Y$-algebras on Y which is flat over S. Let B be a coherent $\mathcal{A}$-module on Y which is flat over S. There is a closed subscheme $\operatorname{Quot}_{B/Y/S}^{\mathcal{A},P}$ of $Q := \operatorname{Quot}_{B/Y/S}^P$ parametrizing all quotient $\mathcal{A}_s$-modules F of B_s with $\chi(F(n)) = P(n)$.*

Proof. Let $\mathcal{Q}$ and $\mathcal{K}$ be the universal quotient and the universal subsheaf of $B \otimes_{\mathcal{O}_S} \mathcal{O}_Q$:

$$0 \to \mathcal{K} \to B \otimes_{\mathcal{O}_S} \mathcal{O}_Q \to \mathcal{Q} \to 0. \tag{1.96}$$

Then we have a homomorphism

$$\mathcal{K} \otimes_{\mathcal{O}_S} \mathcal{A} \to B \otimes_{\mathcal{O}_S} \mathcal{O}_Q \otimes_{\mathcal{O}_S} \mathcal{A} \to B \otimes_{\mathcal{O}_S} \mathcal{O}_Q \to \mathcal{Q} \tag{1.97}$$

induced by the multiplication map $B \otimes_{\mathcal{O}_S} \mathcal{O}_Q \otimes_{\mathcal{O}_S} \mathcal{A} \to B \otimes_{\mathcal{O}_S} \mathcal{O}_Q$. Let $Z = \operatorname{Quot}^{\mathcal{A},P}_{B/Y/S}$ be the zero locus of this homomorphism. Then for an S-morphism $T \to Q$, $\mathcal{K} \otimes_{\mathcal{O}_S} \mathcal{O}_T$ is an $\mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{O}_T$-submodule of $B \otimes_{\mathcal{O}_S} \mathcal{O}_T$ if and only if $T \to Q$ factors through Z. □

Corollary 1.3.8. *Let G' be a family of objects in $\mathcal{C}_s$, $s \in S$. Then there is a quot-scheme $\operatorname{Quot}^{\mathcal{C},P}_{G'/X/S}$ parametrizing all quotients $G'_s \to E$ in $\mathcal{C}_s$, where P is the G_s-twisted Hilbert-polynomial of the quotient $G_s \to E$, $s \in S$.*

Proof. We set $\mathcal{A} := \pi_*(G^\vee \otimes_{\mathcal{O}_X} G)$. Then $\mathcal{A}$ is a flat family of $\mathcal{O}_Y$-algebras on Y and we have an equivalence between the category of $\mathcal{A}_T$-modules F flat over T and the categoty of families E of objects in $\mathcal{C}_t, t \in T$ by $F \mapsto \pi_T^{-1}(F) \otimes_{\pi^{-1}(\mathcal{A}_T)} G_T$. So the claim holds. □

1.4. **Stability for perverse coherent sheaves.** For a non-zero object $E \in \mathcal{C}_s$, $\chi(G_s, E(n)) = \chi(\mathbf{R}\pi_*(G_s^\vee \otimes E)(n)) > 0$ for $n \gg 0$ and there are integers $a_i(E)$ such that

$$\chi(G_s, E(n)) = \sum_i a_i(E) \binom{n+i}{i}. \tag{1.98}$$

Definition 1.4.1 (Simpson)**.** Assume that $\mathcal{C}_s$ is a tilting of $\operatorname{Coh}(X_s)$ for all $s \in S$.

(1) An object $E \in \mathcal{C}_s$ is d-dimensional, if $a_d(E) > 0$ and $a_i(E) = 0$, $i > d$.

(2) An object $E \in \mathcal{C}_s$ of dimension d is G_s-twisted semi-stable if

$$\chi(G_s, F(n)) \le \frac{a_d(F)}{a_d(E)} \chi(G_s, E(n)), n \gg 0 \tag{1.99}$$

for all proper subobject F of E.

Remark 1.4.2. (1) If $\dim E > \dim \pi(Z_s)$ and E is G_s-twisted semi-stable, then $H^{-1}(E) = 0$. Indeed $H^{-1}(E)[1]$ is a subobject of E with

$$\deg \chi(G_s, H^{-1}(E)(n)) \le \dim \pi(Z_s) < \deg \chi(G_s, E(n)). \tag{1.100}$$

(2) Assume that $E \in \operatorname{Coh}(X_s) \cap \mathcal{C}_s$. For an exact sequence

$$0 \to F \to E \to F' \to 0 \tag{1.101}$$

in $\mathcal{C}_s$, we have an exact sequence in $\operatorname{Coh}(X_s)$

$$H^{-1}(F') \xrightarrow{\varphi} H^0(F) \to H^0(E) \to H^0(F') \to 0. \tag{1.102}$$

Since $\chi(G_s, H^0(F)(n)) \le \chi(G_s, (\operatorname{coker} \varphi)(n))$, in order to check the semi-stability of E, we may assume that $H^{-1}(F') = 0$.

Proposition 1.4.3. *There is a coarse moduli scheme $\overline{M}^{\mathcal{C},P}_{X/S} \to S$ of G_s-twisted semi-stable objects $E \in \mathcal{C}_s$ with the G_s-twisted Hilbert polynomial P. $\overline{M}^{\mathcal{C},P}_{X/S}$ is a projective scheme over S.*

Proof. The claim is due to Simpson [S, Thm. 4.7]. We set $\mathcal{A} := \pi_*(G^\vee \otimes G)$. If we set $\Lambda_0 = \mathcal{O}_Y$ and $\Lambda_k = \mathcal{A}$ for $k \ge 1$, then a sheaf of $\mathcal{A}$-module is an example of Λ-modules in [S]. Let Q^{ss} be an open subscheme of $\operatorname{Quot}^{\mathcal{A},P}_{\mathcal{A}(-n)\otimes V/Y/S}$ consisting of semi-stable $\mathcal{A}_s$-modules on Y_s, $s \in S$. Then we have the moduli space $\overline{M}^{\mathcal{A},P}_{Y/S} \to S$ of semi-stable $\mathcal{A}_s$-modules on Y_s as a GIT-quotient $Q^{ss} /\!\!/ GL(V)$, where we use a natural polarization on the embedding of the quot-scheme into the Grassmannian. By a standard argument due to Langton, we see that $\overline{M}^{\mathcal{A},P}_{Y/S}$ is projective over S. Since the semi-stable $\mathcal{A}_s$-modules correspond to G_s-twisted semi-stable objects via the Morita equivalence (Proposition 1.3.5), we get the moduli space $\overline{M}^{\mathcal{C},P}_{X/S} \to S$, which is projective over S. □

We consider a natural relative polarization on $\overline{M}^{\mathcal{C},P}_{X/S}$. Let Q^{ss} be the open subscheme of $\operatorname{Quot}^{\mathcal{C},P}_{G(-n)\otimes V/X/S} \cong \operatorname{Quot}^{\mathcal{A},P}_{\mathcal{A}(-n)\otimes V/Y/S}$ such that $\overline{M}^{\mathcal{C},P}_{X/S} = Q^{ss} /\!\!/ GL(V)$, where V is a vector space of dimension $P(n)$. Let $\mathcal{Q}$ be the universal quotient on $Q^{ss} \times X$. Then $\mathcal{Q}_{|\{q\}\times X}$ is G-twisted semi-stable for all $q \in Q^{ss}$. By the construction of the moduli space, we have a $GL(V)$-equivariant isomorphism $V \to p_{Q^{ss}}(G^\vee \otimes \mathcal{Q}(n))$. We set

$$\begin{aligned} \mathcal{L}_{m,n} &:= \det p_{Q^{ss}!}(G^\vee \otimes \mathcal{Q}(n+m))^{\otimes P(n)} \otimes \det p_{Q^{ss}!}(G^\vee \otimes \mathcal{Q}(n))^{\otimes(-P(m+n))} \\ &= \det p_{Q^{ss}!}(G^\vee \otimes \mathcal{Q}(n+m))^{\otimes P(n)} \otimes \det V^{\otimes(-P(m+n))}. \end{aligned} \tag{1.103}$$

We note that $\mathbf{R}\pi_*(G^\vee \otimes \mathcal{Q})$ gives the universal quotient $\mathcal{A}$-module on $Y \times \mathrm{Quot}^{\mathcal{A},P}_{\mathcal{A}(-n)\otimes V/Y/S}$. By the construction of the moduli space, we get the following.

Lemma 1.4.4. *$\mathcal{L}_{m,n}$, $m \gg n \gg 0$ is the pull-back of a relatively ample line bundle on $\overline{M}^{\mathcal{C},P}_{X/S}$.*

Assume that $S = \mathrm{Spec}(\mathbb{C})$ and $\dim X = 2$. We set $\mathcal{O}_X(1) = \mathcal{O}_X(H)$.

Definition 1.4.5. (1) For $\mathbf{e} \in K(X)_{\mathrm{top}}$, $\overline{M}^G_H(\mathbf{e})$ is the moduli space of G-twisted semi-stable objects E of $\mathcal{C}$ with $\tau(E) = \mathbf{e}$ and $M^G_H(\mathbf{e})$ the open subscheme consisting of G-twisted stable objects.

(2) Let $\mathcal{M}_H(\mathbf{e})^{\mu\text{-ss}}$ (resp. $\mathcal{M}^G_H(\mathbf{e})^{ss}, \mathcal{M}^G_H(\mathbf{e})^s$) be the moduli stack of μ-semi-stable (resp. G-twisted semi-stable, G-twisted stable) objects E of $\mathcal{C}$ with $\tau(E) = \mathbf{e}$.

We set $r_0 := \mathrm{rk}\,\mathbf{e}$ and $\xi_0 := c_1(\mathbf{e})$. Then we see that

$$\begin{aligned}
&\mathrm{ch}(P(n)G^\vee((n+m)H) - P(n+m)G^\vee(nH)) \\
=&m\left[\frac{(\mathrm{rk}\,G)r_0}{2}(H^2)\left\{(m-2n)\,\mathrm{ch}\,G^\vee - n(n+m)((\mathrm{rk}\,G)H - (c_1(G),H)\varrho_X)\right\}\right. \\
&\left.+(H, (\mathrm{rk}\,G)\xi_0 - r_0 c_1(G) - \frac{(\mathrm{rk}\,G)r_0}{2}K_X)\left(-\,\mathrm{ch}\,G^\vee + \frac{n(n+m)}{2}(H^2)(\mathrm{rk}\,G)\varrho_X\right)\right].
\end{aligned} \tag{1.104}$$

Lemma 1.4.6. *We take $\zeta \in K(X)$ with $\mathrm{ch}(\zeta) = r_0 H + (\xi_0, H)\varrho_X$. Assume that $\tau(G) \in \mathbb{Z}\mathbf{e}$. If $\chi(\mathbf{e},\mathbf{e}) = 0$ and $E \cong E \otimes K_X$ for all $E \in \mathcal{M}^G_H(\mathbf{e})^{ss}$, then $\det p_{Q^{ss}!}(\mathcal{Q} \otimes \zeta^\vee) \cong \det p_{Q^{ss}!}(\mathcal{Q}^\vee \otimes \zeta)^\vee$ is the pull-back of an ample line bundle $\mathcal{L}(\zeta)$ on $\overline{M}^G_H(\mathbf{e})$.*

Proof. We first note that $\det p_{Q^{ss}!}(\mathcal{Q} \otimes E^\vee) \cong \mathcal{O}_{Q^{ss}}$ for $E \in \mathcal{M}^G_H(\mathbf{e})^{ss}$. We set $\tau(G) = \lambda\mathbf{e}$, $\lambda \in \mathbb{Z}_{>0}$. Then $P(n)G^\vee((n+m)H) - P(n+m)G^\vee(nH) \equiv mn(n+m)\lambda\zeta^\vee \mod \mathbb{Z}\mathbf{e}^\vee$. By Lemma 1.4.4, we get our claim. □

Definition 1.4.7. (1) $P(\mathbf{e})$ is the set of subobject E' of $E \in \mathcal{M}_H(\mathbf{e})^{\mu\text{-ss}}$ such that

$$\frac{(c_1(G^\vee \otimes E), H)}{\mathrm{rk}\,E} = \frac{(c_1(G^\vee \otimes E'), H)}{\mathrm{rk}\,E'}. \tag{1.105}$$

(2) For $E' \in P(\mathbf{e})$, we define a wall $W_{E'} \subset \mathrm{NS}(X) \otimes \mathbb{R}$ as the set of $\alpha \in \mathrm{NS}(X) \otimes \mathbb{R}$ satisfying

$$\left(\alpha, \frac{c_1(G^\vee \otimes E)}{\mathrm{rk}\,E} - \frac{c_1(G^\vee \otimes E')}{\mathrm{rk}\,E'}\right) + \left(\frac{\chi(G^\vee \otimes E)}{\mathrm{rk}\,E} - \frac{\chi(G^\vee \otimes E')}{\mathrm{rk}\,E'}\right) = 0. \tag{1.106}$$

Since $\tau(E')$ is finite, $\cup_{E'} W_{E'}$ is locally finite. If $\alpha \in \mathrm{NS}(X) \otimes \mathbb{Q}$ does not lie on any $W_{E'}$, we say that α is general. If a local projective generator G' satisfies $\alpha := c_1(G')/\mathrm{rk}\,G' - c_1(G)/\mathrm{rk}\,G \notin \cup_{E'} W_{E'}$, then we also call G' is general.

Lemma 1.4.8. *If G is general, i.e., $0 \notin \cup_{E'} W_{E'}$, then for $E' \in P(\mathbf{e})$,*

$$\frac{\chi(G,\mathbf{e})}{\mathrm{rk}\,\mathbf{e}} = \frac{\chi(G,E')}{\mathrm{rk}\,E'} \iff \frac{\mathbf{e}}{\mathrm{rk}\,\mathbf{e}} = \frac{\tau(E')}{\mathrm{rk}\,E'} \in K(X)_{\mathrm{top}} \otimes \mathbb{Q}. \tag{1.107}$$

In particular, if $\mathbf{e}$ is primitive, then $\overline{M}^G_H(\mathbf{e}) = M^G_H(\mathbf{e})$ for a general G.

1.5. **A generalization of stability for 0-dimensional objects.** It is easy to see that every 0-dimensional object is G_s-twisted semi-stable. Our definition is not sufficient in order to get a good moduli space. So we introduce a refined version of twisted stability.

Definition 1.5.1. Let G, G' be families of local projective generators of $\mathcal{C}_s$. A 0-dimensional object E is (G_s, G'_s)-twisted semi-stable, if

$$\frac{\chi(G'_s, E_1)}{\chi(G_s, E_1)} \le \frac{\chi(G'_s, E)}{\chi(G_s, E)} \tag{1.108}$$

for all proper subobject E_1 of E.

By a modification of Simpson's construction of moduli spaces, we can construct the coarse moduli scheme of (G_s, G'_s)-twisted semi-stable objects. From now on, we assume that $S = \mathrm{Spec}(\mathbb{C})$ for simplicity.

Lemma 1.5.2. *Let G be a locally free sheaf on X which is a local projective generator of $\mathcal{C}$.*

(1) *Assume that there is an exact sequence in $\mathcal{C}$*

$$0 \to E' \to V_0 \to V_1 \to \cdots \to V_r \to E \to 0 \tag{1.109}$$

such that V_i are local projective objects of $\mathcal{C}$. If $r \ge \dim X$, then E' is a local projective object of $\mathcal{C}$.

(2) *For $E \in K(Y)$, there is a local projective generator G' of $\mathcal{C}$ such that $E = G' - NG(-n)$, where N and n are sufficiently large integers.*

Proof. (1) We first prove that $H^i(\mathbf{R}\pi_*\mathbf{R}\mathcal{H}om(E,F))=0$, $i>\dim X+1$ for all $F\in\mathcal{C}$. Since $\mathcal{C}$ is a tilting of $\operatorname{Coh}(X)$ (Lemma 1.1.7), $H^i(E)=H^i(F)=0$ for $i\neq -1,0$. By using a spectral sequence, we get

$$H^i(\mathbf{R}\pi_*\mathbf{R}\mathcal{H}om(H^{-p}(E)[p],H^{-q}(F)[q]))=0 \tag{1.110}$$

for $i>\dim X+1$. Hence we get $H^i(\mathbf{R}\pi_*\mathbf{R}\mathcal{H}om(E,F))=0$, $i>\dim X+1$. Then we see that

$$H^i(\mathbf{R}\pi_*\mathbf{R}\mathcal{H}om(E',F))\cong H^{i+r+1}(\mathbf{R}\pi_*\mathbf{R}\mathcal{H}om(E,F))=0 \tag{1.111}$$

for all integer with $i>\max\{\dim X-r,0\}=0$. Therefore E' is a local projective object.

(2) We first prove that there are local projective generators G_1,G_2 such that $E=G_1-G_2$. We may assume that $E\in\mathcal{C}$. We take a resolution of E

$$0\to E'\to G(-n_r)^{\oplus N_r}\overset{\phi}{\to} G(-n_{r-1})^{\oplus N_{r-1}}\to\cdots\to G(-n_0)^{\oplus N_0}\to E\to 0. \tag{1.112}$$

If $r\geq\dim X$, then (1) implies that E' is a local projective object. We set $r:=2j_0+1$. We set $G_1:=E'\oplus\oplus_{j=0}^{j_0}G(-n_{2j})^{\oplus N_{2j}}$ and $G_2:=\oplus_{j=0}^{j_0}G(-n_{2j+1})^{\oplus N_{2j+1}}$ Then G_1 and G_2 are local projective generators and $E=G_1-G_2$. We take a resolution

$$0\to G_2'\to G(-n)^{\oplus N}\to G_2\to 0 \tag{1.113}$$

such that $G_2'\in\mathcal{C}$. Then we see that $\mathbf{R}\pi_*\mathbf{R}\mathcal{H}om(G_2',F)\in\operatorname{Coh}(Y)$ for any $F\in\mathcal{C}$. Since $E=(G_1\oplus G_2')-G(-n)^{\oplus N}$ and $G_1\oplus G_2'$ is a local projective generator, we get our claim. □

Definition 1.5.3. Let A be an element of $K(Y)\otimes\mathbb{Q}$ and G a local projective generator. A 0-dimensional object E is (G,A)-twisted semi-stable, if

$$\frac{\chi(A,F)}{\chi(G,F)}\leq\frac{\chi(A,E)}{\chi(G,E)} \tag{1.114}$$

for all proper subobject F of E.

By Lemma 1.5.2, we write $N'A=G'-NG(-n)\in K(X)$, where G' is a local projective generator and $n,N,N'>0$. Then

$$\frac{\chi(G',E)}{\chi(G,E)}=N'\frac{\chi(A,E)}{\chi(G,E)}+N. \tag{1.115}$$

Hence E is (G,G')-twisted semi-stable if and only if E is (G,A)-twisted semi-stable. Thus we get the following proposition.

Proposition 1.5.4. *Let A be an element of $K(Y)\otimes\mathbb{Q}$ and G a local projective generator. Let v be a Mukai vector of a 0-dimensional object.*

1. *There is a coarse moduli scheme $\overline{M}_{\mathcal{O}_X(1)}^{G,A}(v)$ of (G,A)-twisted semi-stable objects of $\mathcal{C}$.*
2. *If v is primitive and A is general in $K(Y)\otimes\mathbb{Q}$, then $\overline{M}_{\mathcal{O}_X(1)}^{G,A}(v)$ consists of (G,A)-twisted stable objects. Moreover $\overline{M}_{\mathcal{O}_X(1)}^{G,A}(\varrho_X)$ is a fine moduli space.*

Remark 1.5.5. If $v(E)=\varrho_X$ and $\operatorname{rk}A=0$, then E is (G,A)-twisted semi-stable if and only if $\chi(A,E')\leq 0$ for all subobject E' of E in $\mathcal{C}$. Thus the semi-stability does not depend on the choice of G.

Remark 1.5.6. In subsection 1.7, we deal with the twisted sheaves. In this case, we still have the moduli spaces of 0-dimensional stable objects, but $\overline{M}_{\mathcal{O}_X(1)}^{G,A}(\varrho_X)$ does not have a universal family.

1.6. **Construction of the moduli spaces of $\mathcal{A}$-modules of dimension 0.** By Proposition 1.1.3, we have an equivalence $\mathcal{C}\to\operatorname{Coh}_{\mathcal{A}}(Y)$. We set $\mathcal{B}:=\pi_*(G^\vee\otimes G')$. Then $\mathcal{B}$ is a local projective generator of $\operatorname{Coh}_{\mathcal{A}}(Y)$: For all $F\in\operatorname{Coh}_{\mathcal{A}}(Y)$, $\mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{B},F)=\mathcal{H}om_{\mathcal{A}}(\mathcal{B},F)$ and $\mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{B},F)=0$ if and only if $F=0$. In particular, we have a surjective morphism

$$\phi:\mathcal{H}om_{\mathcal{A}}(\mathcal{B},F)\otimes_{\mathcal{A}}\mathcal{B}\to F. \tag{1.116}$$

For $F\in\operatorname{Coh}_{\mathcal{A}}(Y)$, we set

$$\chi_{\mathcal{A}}(\mathcal{B},F):=\chi(\mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{B},F)). \tag{1.117}$$

For $F\in\operatorname{Coh}_{\mathcal{A}}(Y)$, $\pi^{-1}(F)\otimes_{\pi^{-1}(\mathcal{A})}G$ is (G,G')-twisted semi-stable, if

$$\frac{\chi_{\mathcal{A}}(\mathcal{B},F_1)}{\chi(F_1)}\leq\frac{\chi_{\mathcal{A}}(\mathcal{B},F)}{\chi(F)} \tag{1.118}$$

for all proper sub $\mathcal{A}$-module F_1 of F. We define the $(\mathcal{A},\mathcal{B})$-twisted semi-stability by this inequality.

Proposition 1.6.1. *There is a coarse moduli scheme of $(\mathcal{A},\mathcal{B})$-twisted semi-stable $\mathcal{A}$-modules of dimension 0.*

Proof of Proposition 1.6.1. Let F be an $\mathcal{A}$-module of dimension 0. Then $\operatorname{Hom}_{\mathcal{A}}(\mathcal{B},F)\otimes\mathcal{B}\to F$ is surjective. Hence all 0-dimensional objects F are parametrized by a quot-scheme $Q:=\operatorname{Quot}^{\mathcal{A},m}_{V\otimes\mathcal{B}/Y/\mathbb{C}}$, where $m=\chi(F)$ and $\dim V=\chi_{\mathcal{A}}(\mathcal{B},F)$. Let $V\otimes\mathcal{O}_Q\otimes\mathcal{B}\to\mathcal{F}$ be the universal quotient. For simplicity, we set $\mathcal{F}_q:=\mathcal{F}_{|\{q\}\times Y}$, $q\in Q$. For a sufficiently large integer n, we have a quotient $V\otimes H^0(Y,\mathcal{B}(n))\to H^0(Y,F(n))$. We set $W:=H^0(Y,\mathcal{B}(n))$. Then we have an embedding

$$\operatorname{Quot}^{\mathcal{A},m}_{V\otimes\mathcal{B}/Y/\mathbb{C}}\hookrightarrow Gr(V\otimes W,m). \tag{1.119}$$

This embedding is equivariant with respect to the natural action of $PGL(V)$. The following is well-known.

Lemma 1.6.2. *Let $\alpha: V\otimes W\to U$ be a point of $\mathfrak{G}:=Gr(V\otimes W,m)$. Then α belongs to the set $\mathfrak{G}^{ss}$ of semi-stable points if and only if*

$$\frac{\dim U}{\dim V}\le\frac{\dim\alpha(V_1\otimes W)}{\dim V_1} \tag{1.120}$$

for all proper subspace $V_1\neq 0$ of V. If the inequality is strict for all V_1, then α is stable.

We set

$$Q^{ss}:=\{q\in Q|\ \mathcal{F}_q \text{ is } (\mathcal{A},\mathcal{B})\text{-twisted semi-stable }\}. \tag{1.121}$$

For $q\in Q^{ss}$, $V\to\operatorname{Hom}_{\mathcal{A}}(\mathcal{B},F)$ is an isomorphism. We only prove that $Q^{ss}=\mathfrak{G}^{ss}\cap Q$. Then Proposition 1.6.1 easily follows.

For an $\mathcal{A}$-submodule F_1 of F, we set $V_1:=\operatorname{Hom}_{\mathcal{A}}(\mathcal{B},F_1)$. Then we have a surjective homomorphism $V_1\otimes\mathcal{B}\to F_1$. Conversely for a subspace V_1 of V, we set $F_1:=\operatorname{im}(V_1\otimes\mathcal{B}\to F)$. Then $V_1\to\operatorname{Hom}_{\mathcal{A}}(\mathcal{B},F_1)$ is injective.

We set

$$\mathfrak{F}:=\{\operatorname{im}(V_1\otimes\mathcal{B}\to\mathcal{F}_q)|q\in Q,\ V_1\subset V\}. \tag{1.122}$$

Since $\mathfrak{F}$ is bounded, we can take an integer n in the definition of W such that $V_1\otimes W\to H^0(Y,F_1)$ is surjective for all $F_1\in\mathfrak{F}$. Assume that $\mathcal{F}_q$ is $(\mathcal{A},\mathcal{B})$-twisted semi-stable. For any $V_1\subset V$, we set $F_1:=\operatorname{im}(V_1\otimes\mathcal{B}\to\mathcal{F}_q)$. Then $\alpha(V_1\otimes W)=H^0(Y,F_1)$. Hence

$$\frac{\dim\alpha(V_1\otimes W)}{\dim V_1}\ge\frac{\chi(F_1)}{\dim\operatorname{Hom}_{\mathcal{A}}(\mathcal{B},F_1)}=\frac{\chi(F_1)}{\chi_{\mathcal{A}}(\mathcal{B},F_1)}\ge\frac{\chi(\mathcal{F}_q)}{\chi_{\mathcal{A}}(\mathcal{B},\mathcal{F}_q)}=\frac{\dim\alpha(V\otimes W)}{\dim V}. \tag{1.123}$$

Thus $q\in\mathfrak{G}^{ss}$.

We take a point $q\in\mathfrak{G}^{ss}\cap Q$. We first prove that $\psi: V\to\operatorname{Hom}_{\mathcal{A}}(\mathcal{B},\mathcal{F}_q)$ is an isomorphism. We set $V_1:=\ker\psi$. Since $V_1\otimes\mathcal{B}\to\mathcal{F}_q$ is 0, we get $\alpha(V_1\otimes W)=0$. Then

$$\frac{\dim U}{\dim V}\le\frac{\dim\alpha(V_1\otimes W)}{\dim V_1}=0, \tag{1.124}$$

which is a contradiction. Therefore ψ is injective. Since $\dim V=\dim\operatorname{Hom}_{\mathcal{A}}(\mathcal{B},\mathcal{F}_q)$, ψ is an isomorphism. Let $F_1\neq 0$ be a proper $\mathcal{A}$-submodule of $\mathcal{F}_q$. We set $V_1:=\operatorname{Hom}_{\mathcal{A}}(\mathcal{B},F_1)$. Then

$$\frac{\chi(F_1)}{\dim\operatorname{Hom}_{\mathcal{A}}(\mathcal{B},F_1)}\ge\frac{\dim\alpha(V_1\otimes W)}{\dim V_1}\ge\frac{\dim\alpha(V\otimes W)}{\dim V}=\frac{\chi(\mathcal{F}_q)}{\chi_{\mathcal{A}}(\mathcal{B},\mathcal{F}_q)}. \tag{1.125}$$

Hence $\mathcal{F}_q$ is $(\mathcal{A},\mathcal{B})$-twisted semi-stable. If q is a stable point, then we also see that $\mathcal{F}_q$ is $(\mathcal{A},\mathcal{B})$-twisted stable.

1.7. Twisted case.

1.7.1. *Definition.* Let $X=\cup_i X_i$ be an analytic open covering of X and $\beta=\{\beta_{ijk}\in H^0(X_i\cap X_j\cap X_k,\mathcal{O}_X^\times)\}$ a Cech 2-cocycle of $\mathcal{O}_X^\times$. We assume that β defines a torsion element $[\beta]$ of $H^2(X,\mathcal{O}_X^\times)$. Let $E=(\{E_i\},\{\varphi_{ij}\})$ be a coherent β-twisted sheaf:

(i) E_i is a coherent sheaf on X_i.
(ii) $\varphi_{ij}: E_{i|X_i\cap X_j}\to E_{j|X_i\cap X_j}$ is an isomorphism.
(iii) $\varphi_{ji}=\varphi_{ij}^{-1}$.
(iv) $\varphi_{ki}\circ\varphi_{jk}\circ\varphi_{ij}=\beta_{ijk}\operatorname{id}_{X_i\cap X_j\cap X_k}$.

Let G be a locally free β-twisted sheaf and $P:=\mathbb{P}(G^\vee)$ the associated projective bundle over X (cf. [Y4, sect. 1.1]). Let $w(P)\in H^2(X,\mathbb{Z}/r\mathbb{Z})$ be the characteristic class of P ([Y4, Defn. 1.2]). Then $[\beta]$ is trivial if and only if $w(P)\in\operatorname{im}(\operatorname{NS}(X)\to H^2(X,\mathbb{Z}/r\mathbb{Z}))$ ([Y4, Lem. 1.4]).

Let $\operatorname{Coh}^\beta(X)$ be the category of coherent β-twisted sheaves on X and $\mathbf{D}^\beta(X)$ the bounded derived category of $\operatorname{Coh}^\beta(X)$. Let $K^\beta(X)$ be the Grothendieck group of $\operatorname{Coh}^\beta(X)$. Then similar statements in Lemma 1.1.5 hold for $\operatorname{Coh}^\beta(X)$. Then all results in sections 1.3 and 1.4 hold. In particular, if a locally free β-twisted sheaf G defines a torsion pair, then we have the moduli of G-twisted semi-stable objects. Replacing $\zeta\in K(X)$ by $\zeta\in K^\beta(X)$ with $c_1(\zeta)=r_0H$ and $\chi(G\otimes\zeta^\vee)=0$, Lemma 1.4.6 also holds.

1.7.2. *Chern character.* We have a homomorphism

$$\begin{array}{rccc} \operatorname{ch}_G : & \mathbf{D}^\beta(X) & \to & H^{ev}(X,\mathbb{Q}) \\ & E & \mapsto & \frac{\operatorname{ch}(G^\vee\otimes E)}{\sqrt{\operatorname{ch}(G^\vee\otimes G)}}. \end{array} \tag{1.126}$$

Obviously $\operatorname{ch}_G(E)$ depends only on the class in $K^\beta(X)$. Since

$$\operatorname{ch}_G(E)^\vee \operatorname{ch}_G(F) = \frac{\operatorname{ch}((G^\vee \otimes E)^\vee \otimes (G^\vee \otimes F))}{\operatorname{ch}(G^\vee \otimes G)} = \operatorname{ch}(E^\vee \otimes F), \tag{1.127}$$

we have the following Riemann-Roch formula.

$$\chi(E,F) = \int_X \operatorname{ch}_G(E)^\vee \operatorname{ch}_G(F) \operatorname{td}_X . \tag{1.128}$$

Assume that X is a surface. For a torsion G-twisted sheaf E, we can attach the codimension 1 part of the scheme-theoretic support $\operatorname{Div}(E)$ as in the usual sheaves. Then we see that

$$\operatorname{ch}_G(E) = (0, [\operatorname{Div}(E)], a), a \in \mathbb{Q}, \tag{1.129}$$

where $[\operatorname{Div}(E)]$ denotes the homology class of the divisor $\operatorname{Div}(E)$ and we regard it as an element of $H^2(X,\mathbb{Z})$ by the Poincaré duality. More generally, if $E \in \mathbf{D}^\beta(X)$ satisfies $\operatorname{rk} H^i(E) = 0$ for all i, then

$$\operatorname{ch}_G(E) = (0, \sum_i (-1)^i [\operatorname{Div}(H^i(E))], a), a \in \mathbb{Q}. \tag{1.130}$$

We set $c_1(E) := \sum_i (-1)^i [\operatorname{Div}(H^i(E))]$.

Remark 1.7.1. If $H^3(X,\mathbb{Z})$ is torsion free, then we have an automorphism η of $H^*(X,\mathbb{Q})$ such that the image of $\eta \circ \operatorname{ch}_G$ is contained in $\operatorname{ch}(K(X)) \subset \mathbb{Z} \oplus H^2(X,\mathbb{Z}) \oplus H^4(X, \frac{1}{2}\mathbb{Z})$ and (1.128) holds if we replace ch_G by $\eta \circ \operatorname{ch}_G$ (cf. [Y4]): We first note that

$$\operatorname{ch}(K(X)) = \{(r, D, a) | r \in \mathbb{Z}, D \in H^2(X,\mathbb{Z}), a - (D, K_X)/2 \in \mathbb{Z}\}. \tag{1.131}$$

Replacing the statement of [Y4, Lem. 3.1] by

$$\begin{aligned} & c_2(E^\vee \otimes E) + r(r-1)(w(E), K_X) \\ \equiv & -(r-1)((w(E)^2) - r(w(E), K_X)) \mod 2r, \end{aligned} \tag{1.132}$$

we can prove a similar claim to [Y4, Lem. 3.3].

Lemma 1.7.2. *Let E be a β-twisted sheaf of* $\operatorname{rk} E = 0$. *Then*

$$[\chi(G,E) \mod r\mathbb{Z}] \equiv -w(P) \cap [\operatorname{Div}(E)], \tag{1.133}$$

where we identified $H_0(X,\mathbb{Z}/r\mathbb{Z})$ with $\mathbb{Z}/r\mathbb{Z}$.

Proof. Since $\chi(G,E)$ and $[\operatorname{Div}(E))]$ are additive, it is sufficient to prove the claim for pure sheaves. If $\dim E = 0$ as an object of $\operatorname{Coh}^\beta(X)$, then $r|\chi(G,E)$ and $\operatorname{Div}(E) = 0$. Hence the claim holds. We assume that E is purely 1-dimensional. Then E is a twisted sheaf on $C := \operatorname{Div}(E)$. Since C is a curve, there is a β-twisted line bundle L on C and we have an equivalence

$$\begin{array}{rccc} \varphi : & \operatorname{Coh}^\beta(C) & \to & \operatorname{Coh}(C) \\ & E & \mapsto & E \otimes L^\vee. \end{array} \tag{1.134}$$

Then we can take a filtration $0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = E$ of E such that $\operatorname{Div}(F_i/F_{i-1})$ are reduced and irreducible curve and F_i/F_{i-1} are torsion free β-twisted sheaves of rank 1 on $\operatorname{Div}(F_i/F_{i-1})$. Replacing E by F_i/F_{i-1}, we may assume that E is a twisted sheaf of rank 1 on an irreducible and reduced curve $C = \operatorname{Div}(E)$. Then $\chi(G,E) = \chi(\varphi(G_{|C})^\vee \otimes \varphi(E)) = \int_C c_1(\varphi(G_{|C})^\vee) + r\chi(\varphi(E))$. Since $w(P)_{|C} = w(P_{|C}) = c_1(\varphi(G_{|C}))$ mod $r\mathbb{Z}$, $[\chi(G,E) \mod r\mathbb{Z}] \equiv -w(P) \cap [C]$. □

Corollary 1.7.3. *For an object E of $\mathbf{D}^\beta(X)$, assume that* $\operatorname{rk} H^i(E) = 0$ *for all i. Then*

$$[\chi(G,E) \mod r\mathbb{Z}] \equiv -w(P) \cap [\operatorname{Div}(E)]. \tag{1.135}$$

Moreover if $c_1(E) = 0$, then $\operatorname{ch}_G(E) \in \mathbb{Z}\varrho_X$.

Proof. The second claim follows from $\int_X \operatorname{ch}_G(E) = \chi(G,E)/r = (\chi(G,E)/r) \int_X \varrho_X$. □

2.1. **Perverse coherent sheaves on the resolution of rational singularities.** Let Y be a projective normal surface with at worst rational singularities and $\pi : X \to Y$ the minimal resolution. Let p_i, $i = 1, 2, ..., n$ be the singular points of Y and $Z_i := \pi^{-1}(p_i) = \sum_{j=1}^{s_i} a_{ij} C_{ij}$ their fundamental cycles. Let β be a 2-cocycle of $\mathcal{O}_X^\times$ whose image in $H^2(X, \mathcal{O}_X^\times)$ is a torsion element. For β-twisted line bundles L_{ij} on C_{ij}, we shall define abelian categories $\operatorname{Per}(X/Y, \{L_{ij}\})$ and $\operatorname{Per}(X/Y, \{L_{ij}\})^*$.

Proposition 2.1.1. (1) *There is a locally free sheaf E such that $\chi(E, L_{ij}) = 0$ for all i, j and $R^1\pi_*(E^\vee \otimes E) = 0$.*

(2) *$\mathcal{C}(E)$ is the tilting of $\operatorname{Coh}^\beta(X)$ with respect to the torsion pair (S, T) such that*

$$S := \{E \in \operatorname{Coh}^\beta(X) | \ E \text{ is generated by subsheaves of } L_{ij} \ \}, \quad T := \{E \in \operatorname{Coh}^\beta(X) | \operatorname{Hom}(E, L_{ij}) = 0\}. \tag{2.1}$$

(3) *$\mathcal{C}(E)^*$ is the tilting of $\operatorname{Coh}^\beta(X)$ with respect to the torsion pair (S^*, T^*) such that*

$$S^* := \{E \in \operatorname{Coh}^\beta(X) | \ E \text{ is generated by subsheaves of } A_{p_i} \otimes \omega_{Z_i} \ \}, \quad T^* := \{E \in \operatorname{Coh}^\beta(X) | \operatorname{Hom}(E, A_{p_i} \otimes \omega_{Z_i}) = 0\}. \tag{2.2}$$

For the proof of (1), we shall use the deformation theory of a coherent twisted sheaf.

Definition 2.1.2. For a coherent β-twisted sheaf E on a scheme W, $\operatorname{Def}(W, E)$ denotes the local deformation space of E fixing $\det E$.

For a complex $E \in \mathbf{D}^\beta(X)$, let

$$\operatorname{Ext}^i(E, E)_0 := \ker(\operatorname{Ext}^i(E, E) \xrightarrow{\operatorname{tr}} H^i(X, \mathcal{O}_X)) \tag{2.3}$$

be the kernel of the trace map. If $\operatorname{Ext}^2(E, E)_0 = 0$, then $\operatorname{Def}(W, E)$ is smooth and the Zariski tangent space at E is $\operatorname{Ext}^1(E, E)_0$. The following is well-known.

Lemma 2.1.3. *Let D be a divisor on X. For $E \in \operatorname{Coh}^\beta(X)$ with $\operatorname{rk} E > 0$, we have a torsion free β-twisted sheaf E' such that $\tau(E') = \tau(E) - n\tau(\mathbb{C}_x)$ and $\operatorname{Ext}^2(E', E'(D))_0 = 0$.*

Proof. For a locally free β-twisted sheaf E, we consider a general surjective homomorphism $\phi : E \to \oplus_{i=1}^n \mathbb{C}_{x_i}$, $x_i \in X$. If n is sufficiently large, then $E' := \ker \phi$ satisfies the claim. □

Lemma 2.1.4. *Let C be an effective divisor on X. For $(r, \mathcal{L}) \in \mathbb{Z}_{>0} \times \operatorname{Pic}(C)$, the moduli stack of locally free sheaves E on C such that $(\operatorname{rk} E, \det E) = (r, \mathcal{L})$ is irreducible.*

Proof. For a locally free sheaf E on C we consider $\phi : H^0(X, E(n)) \otimes \mathcal{O}_C(-n) \to E$. Assume that ϕ is surjective. Then there is a subvector space $V \subset H^0(X, E(n))$ of $\dim V = r-1$ such that $\psi : V \otimes \mathcal{O}_C(-n) \to E$ is injective for any point of C. Then $\operatorname{coker} \psi$ is a line bundle which is isomorphic to $\det(E) \otimes \mathcal{O}_C((r-1)n)$. Hence E is parametrized an affine space $\operatorname{Ext}^1_{\mathcal{O}_C}(\mathcal{L} \otimes \mathcal{O}_C((r-1)n), \mathcal{O}_C(-n) \otimes V) = H^1(C, \mathcal{L}^\vee(-rn) \otimes V)$. Since the surjectivity of ϕ is an open condition and ϕ is surjective for $n \gg 0$, we get our claim. □

Proof of Proposition 2.1.1. (1) For a locally free β-twisted sheaf G on X, we set $g_{ij} := \chi(G, L_{ij})$. Let $\alpha \in \oplus_{i=1}^n \oplus_{j=1}^{s_i} \mathbb{Q}[C_{ij}]$ be a $\mathbb{Q}$-divisor such that $\operatorname{rk} G(\alpha, C_{ij}) = g_{ij}$. We take a locally free sheaf $A \in \operatorname{Coh}(X)$ such that $c_1(A)/\operatorname{rk} A = \alpha$. Then $\chi(G \otimes A, L_{ij}) = \operatorname{rk} A(g_{ij} - \operatorname{rk} G(\alpha, C_{ij})) = 0$ for all i, j. By Lemma 2.1.3, there is a torsion free β-twisted sheaf E on X such that $\tau(E) = \tau(G \otimes A) - n\tau(\mathbb{C}_x)$ and $\operatorname{Hom}(E, E(K_X + C_{ij}))_0 = 0$ for all i, j. We consider the restriction morphism

$$\phi_{ij} : \operatorname{Def}(X, E) \to \operatorname{Def}(C_{ij}, E_{|C_{ij}}). \tag{2.4}$$

Since $\operatorname{Ext}^2(E, E(-C_{ij}))_0 = 0$, we get $\operatorname{Ext}^2(E, E)_0 = 0$. Thus $\operatorname{Def}(X, E)$ is smooth. We also have the smoothness of $\operatorname{Def}(C_{ij}, E_{|C_{ij}})$, by the locally freeness of $E_{|C_{ij}}$. We consider the homomorphism of the tangent spaces

$$\operatorname{Ext}^1_{\mathcal{O}_X}(E, E)_0 \to \operatorname{Ext}^1_{\mathcal{O}_{C_{ij}}}(E_{|C_{ij}}, E_{|C_{ij}})_0. \tag{2.5}$$

Then it is surjective by $\operatorname{Ext}^2(E, E(-C_{ij}))_0 = 0$. Therefore ϕ is submersive. By the equivalence $\varphi : \operatorname{Coh}^\beta(C_{ij}) \to \operatorname{Coh}(C_{ij})$ in (1.134), we have an isomorphism $\operatorname{Def}(C_{ij}, E_{|C_{ij}}) \to \operatorname{Def}(C_{ij}, \varphi(E_{|C_{ij}}))$. Since $\chi(E, L_{ij}) = 0$, $\det(E_{|C_{ij}} \otimes L_{ij}^\vee) = \mathcal{O}_{C_{ij}}(\operatorname{rk} E)$. Then Lemma 2.1.4 implies that E deforms to a β-twisted sheaf such that $E_{|C_{ij}} \cong L_{ij}(1)^{\oplus \operatorname{rk} E}$. Since these conditions are open condition, there is a locally free β-twisted sheaf E such that $E_{|C_{ij}} \cong L_{ij}(1)^{\oplus \operatorname{rk} E}$ for all i, j. By taking the double dual of E and using Lemma 1.2.9, we get (1).

(2) Note that $L_{ij} = A_{p_i} \otimes \mathcal{O}_{C_{ij}}(-1)$. By Proposition 1.2.19 and Proposition 1.1.19, we get the claim. For (3), we use Proposition 1.2.21 and Proposition 1.1.19. □

Definition 2.1.5. (1) We set $\operatorname{Per}(X/Y,\{L_{ij}\}) := \mathcal{C}(E)$ and $\operatorname{Per}(X/Y,\{L_{ij}\})^* := \mathcal{C}(E)^*$.
(2) If β is trivial, then we can write $L_{ij} = \mathcal{O}_{C_{ij}}(b_{ij})$. In this case, we set $\operatorname{Per}(X/Y,\mathbf{b}_1,\dots,\mathbf{b}_n) := \operatorname{Per}(X/Y,\{L_{ij}\})$ and $\operatorname{Per}(X/Y,\mathbf{b}_1,\dots,\mathbf{b}_n)^* := \operatorname{Per}(X/Y,\{L_{ij}\})^*$, where $\mathbf{b}_i := (b_{i1}, b_{i2},\dots,b_{is_i})$.

Remark 2.1.6. If $\mathbf{b}_i(0) = (-1,-1,\dots,-1)$, then $\operatorname{Per}(X/Y,\mathbf{b}_1(0),...,\mathbf{b}_n(0)) = {}^{-1}\operatorname{Per}(X/Y)$.

Definition 2.1.7. We set

$$\begin{aligned} A_0(\mathbf{b}_i) :=& A_{p_i},\\ A_0(\mathbf{b}_i)^* :=& A_{p_i}\otimes \omega_{Z_i}. \end{aligned} \tag{2.6}$$

We collect easy facts on $A_0(\mathbf{b}_i)$ and $A_0(\mathbf{b}_i)^*$ which follow from Lemma 1.2.18 and Lemma 1.2.22.

Lemma 2.1.8. (1) (a) *For $E = A_0(\mathbf{b}_i)$, we have*

$$\operatorname{Hom}(E,\mathcal{O}_{C_{ij}}(b_{ij})) = \operatorname{Ext}^1(E,\mathcal{O}_{C_{ij}}(b_{ij})) = 0,\ 1\le j\le s_i \tag{2.7}$$

and there is an exact sequence

$$0 \longrightarrow F \longrightarrow E \longrightarrow \mathbb{C}_x \longrightarrow 0 \tag{2.8}$$

such that F is a successive extension of $\mathcal{O}_{C_{ij}}(b_{ij})$ and $x\in Z_i$.
(b) *Conversely if E satisfies these conditions, then $E\cong A_0(\mathbf{b}_i)$.*
(2) (a) *For $E = A_0(\mathbf{b}_i)^*$, we have*

$$\operatorname{Hom}(\mathcal{O}_{C_{ij}}(b_{ij}),E) = \operatorname{Ext}^1(\mathcal{O}_{C_{ij}}(b_{ij}),E) = 0,\ 1\le j\le s_i \tag{2.9}$$

and there is an exact sequence

$$0 \longrightarrow E \longrightarrow F \longrightarrow \mathbb{C}_x \longrightarrow 0 \tag{2.10}$$

such that F is a successive extension of $\mathcal{O}_{C_{ij}}(b_{ij})$ and $x\in Z_i$.
(b) *Conversely if E satisfies these conditions, then $E\cong A_0(\mathbf{b}_i)^*$.*

2.2. **Moduli spaces of 0-dimensional objects.** Let $\pi : X\to Y$ be the minimal resolution of a normal projective surface Y and $p_1,p_2,\dots,p_n$ the rational double points of Y as in 2.1. We set $Z := \cup_i Z_i$. Let G be a locally free sheaf on X which is a tilting generator of the category $\mathcal{C} := \mathcal{C}_G$ in Lemma 1.1.5. For $\alpha\in \operatorname{NS}(X)\otimes\mathbb{Q}$, we define α-twisted semi-stability as $\gamma^{-1}((0,\alpha,0))$-twisted stability, where γ is the homomorphism (0.2). In this subsection, we shall study the moduli of α-twisted semi-stable objects. For simplicity, we say that α-twisted semi-stability as α-semi-stability. For simplicity, we set $X^\alpha := \overline{M}_{\mathcal{O}_X(1)}^{G,\alpha}(\varrho_X)$. Since every 0-dimensional object is 0-semi-stable, we have a natural morphism $\pi_\alpha : X^\alpha \to X^0$.

Lemma 2.2.1. *For a 0-dimensional object E of $\mathcal{C}$, there is a proper subspace $T(E)$ of $\operatorname{Ext}^2(E,E)$ such that all obstructions for infinitesimal deformations of E belong to $T(E)$.*

Proof. Let E be a 0-dimensional object of $\mathcal{C}$. We first assume that there is a curve $C\in|K_X|$ such that $C\cap\operatorname{Supp}(E)=\emptyset$. Then $H^0(X,K_X)\to\operatorname{Hom}(E,E(K_X))$ is non-trivial, which implies that the trace map

$$\operatorname{tr}:\operatorname{Ext}^2(E,E)\to H^2(X,\mathcal{O}_X), \tag{2.11}$$

is non-trivial. Since the obstruction for infinitesimal deformations of E lives in $\ker\operatorname{tr}$, $T(E)\subset\ker\operatorname{tr}$ is a proper subspace of $\operatorname{Ext}^2(E,E)$. For a general case, we use the covering trick. Let D be a very ample divisor on Y such that there is a smooth curve $B\in|2D|$ with $B\cap\pi(\operatorname{Supp}(E)\cup Z)=\emptyset$ and $|K_Y+D|$ contains a curve C with $C\cap\pi(\operatorname{Supp}(E)\cup Z)=\emptyset$. Since π is isomorphic over $Y\setminus\pi(Z)$, we may regard B and C as divisors on X. Let $\phi:\widetilde{Y}\to Y$ be the double covering branced along B and set $\widetilde{X}=X\times_Y\widetilde{Y}$. We also denote $\widetilde{X}\to X$ by ϕ. Then $|K_{\widetilde{X}}|=|\phi^*(K_X+D)|$ contains $\phi^*(C)$. Since ϕ is étale over $Y\setminus B$, we have a decomposition $\pi^*(E)=E_1\oplus E_2$ and $\operatorname{Ext}^2(E,E)\to\operatorname{Ext}^2(E_i,E_i)$ are isomorphism for $i=1,2$. Under these isomorphisms, $T(E)$ is mapped into $T(E_i)$. Since $\operatorname{tr}_i:\operatorname{Ext}^2(E_i,E_i)\to H^2(\widetilde{X},\mathcal{O}_{\widetilde{X}})$ are non-trivial, $\ker\operatorname{tr}_i$ are proper subspaces of $\operatorname{Ext}^2(E_i,E_i)$. Hence $T(E)$ is a proper subspace of $\operatorname{Ext}^2(E,E)$. □

Proposition 2.2.2. (1) *For a 0-dimensional object E of $\mathcal{C}$, $E\otimes K_X\cong E$. In particular, $\operatorname{Ext}^2(E,E)\cong\operatorname{Hom}(E,E)^\vee$.*
(2) *For a 0-dimensional Mukai vector v, $M_{\mathcal{O}_X(1)}^{G,\alpha}(v)$ is smooth of dimension $\langle v^2\rangle+2$.*

Proof. (1) Since $K_X=\pi^*(K_Y)$ and $\dim\pi(\operatorname{Supp}(E))=0$, we get $E\otimes K_X\cong E$. (2) For $E\in M_{\mathcal{O}_X(1)}^{G,\alpha}(v)$, we have $\operatorname{Hom}(E,E)=\mathbb{C}$. Then Lemma 2.2.1 implies that $T(E)=0$. Since $\dim\operatorname{Ext}^1(E,E)=\langle v^2\rangle+2$, $M_{\mathcal{O}_X(1)}^{G,\alpha}(v)$ is smooth of dimension $\langle v^2\rangle+2$. □

Remark 2.2.3. There is another argument to prove the smoothness due to Bridgeland [Br1]. We shall use the argument later. So for stable objects, we do not need Lemma 2.2.1, but it is necessary for the study of properly semi-stable objects (see Proposition 2.2.7).

Lemma 2.2.4. *Assume that $\alpha \in \mathrm{NS}(X) \otimes \mathbb{Q}$ satisfies that*

$$(\alpha, D) \neq 0 \text{ for all } D \in \mathrm{NS}(X) \text{ with } (D^2) = -2 \text{ and } (c_1(\mathcal{O}_X(1)), D) = 0. \tag{2.12}$$

Then $X^\alpha = M^{G,\alpha}_{\mathcal{O}_X(1)}(\varrho_X)$.

Proof. Assume that $E \in X^\alpha$ is S-equivalent to $\oplus_{i=1}^t E_i$, where E_i are α-stable objects. Then $(\alpha, c_1(E_i)) = 0$, $(c_1(\mathcal{O}_X(1)), c_1(E_i)) = 0$ and $(c_1(E_i)^2) = \langle v(E_i)^2 \rangle \geq -2$ for all i. Since $\langle v(E_i), v(E_j) \rangle \geq 0$ for $E_i \not\cong E_j$ and $\sum_{i,j} \langle v(E_i), v(E_j) \rangle = \langle v(E)^2 \rangle = 0$, (i) $\langle v(E_i)^2 \rangle = -2$ for an i, or (ii) $\langle v(E_i)^2 \rangle = 0$ for all i. By our choice of α, the case (i) does not occur. In the second case, we see that $v(E_i) = a_i \varrho_X$, $a_i > 0$. Then $\varrho_X = (\sum_i a_i)\varrho_X$, which implies $t = 1$. Therefore E is α-stable. □

Lemma 2.2.5. *Let $\mathcal{E}$ be an object of $\mathbf{D}(X \times X')$ such that $\Phi^{\mathcal{E}^\vee}_{X \to X'} : \mathbf{D}(X) \to \mathbf{D}(X')$ is an equivalence, $\mathcal{E}_{|X \times \{x'\}} \in \mathcal{C}$ for all $x' \in X'$ and $v(\mathcal{E}_{|X \times \{x'\}}) = \varrho_X$. Then every irreducible object of $\mathcal{C}$ appears as a direct summand of the S-equivalence class of $\mathcal{E}_{|X \times \{x'\}}$.*

Proof. Let E be an irreducible object of $\mathcal{C}$. If $\mathrm{Supp}(E) \not\subset Z$, then we have a non-trivial morphism $E \to \mathbb{C}_x$, $x \notin Z$. Since $(\mathcal{C})_{|X \setminus Z} = \mathrm{Coh}(X \setminus Z)$, $\mathbb{C}_x$ is an irreducible object. Hence $E \cong \mathbb{C}_x$. Since $\chi(\mathcal{E}_{|X \times \{x'\}}, \mathbb{C}_x) = 0$ and $\Phi^{\mathcal{E}^\vee}_{X \to X'}$ is an equivalence, there is a point $x' \in X'$ such that $\mathrm{Hom}(\mathcal{E}_{|X \times \{x'\}}, \mathbb{C}_x) \neq 0$ or $\mathrm{Hom}(\mathbb{C}_x, \mathcal{E}_{|X \times \{x'\}}) \neq 0$. Since $v(\mathbb{C}_x) = v(\mathcal{E}_{|X \times \{x'\}}) = \varrho_X$, we get $\mathbb{C}_x \cong \mathcal{E}_{|X \times \{x'\}}$. If $\mathrm{Supp}(E) \subset \cup_i Z_i$, then we still have $\chi(\mathcal{E}_{|X \times \{x'\}}, E) = 0$, since $\mathcal{E}_{|X \times \{x'\}} = \mathbb{C}_x$, $x \notin Z$ for a point $x' \in X'$. Then we have $\mathrm{Hom}(\mathcal{E}_{|X \times \{x'\}}, E) \neq 0$ or $\mathrm{Hom}(E, \mathcal{E}_{|X \times \{x'\}}) \neq 0$. Therefore our claim holds. □

Lemma 2.2.6. *If α is general, then X^α is irreducible.*

Proof. Let X' be a connected component of X^α. Then we have an equivalence $\Phi^{\mathcal{E}^\vee}_{X \to X'} : \mathbf{D}(X) \to \mathbf{D}(X')$, where $\mathcal{E}$ is the universal family. By the same argument as in the proof of Lemma 2.2.5, we see that every $E \in X^\alpha$ belongs to X'. □

Proposition 2.2.7. *Let $\mathcal{X}^0$ be the moduli stack of 0-semi-stable objects E with $v(E) = \rho_X$. Then $\mathcal{X}^0$ is a locally complete intersection stack of dimension 1 and irreducible. In particlar $\mathcal{X}^0$ is a reduced stack.*

Proof. Let Q be an open subscheme of a perverse quot-scheme such that X^0 is a GIT-quotient of a suitable $GL(N)$-action. Then $\mathcal{X}^0$ is the quotient stack $[Q/GL(N)]$. Let $\mathcal{E}$ be the family of 0-dimensional objects of $\mathcal{C}$ on $Q \times X$. For any point $q \in Q$, we set $n_1 := \dim \mathrm{Hom}(\mathcal{K}_q, \mathcal{E}_q)$ and $n_2 := \dim T(\mathcal{E}_q)$, where $\mathcal{K}$ is the universal subobject on $Q \times X$. Then an analytic neighborhood of Q is an intersection of n_2 hypersurfaces in $\mathbb{C}^{n_1}$. Hence $\dim Q \geq n_1 - n_2$ and $\dim[Q/GL(N)] \geq -\chi(\mathcal{E}_q, \mathcal{E}_q) + 1 = 1$. We take a general α and set $Q^u := \{q \in Q | \ \mathcal{E}_q \text{ is not } \alpha\text{-semi-stable}\}$. By the proof of [O-Y, Prop. 2.16], we see that $\dim[Q^u/GL(N)] = 0$. Since $[(Q \setminus Q^u)/GL(N)]$ is the moduli stack of α-stable objects, it is a smooth and irreducible stack of dimension 1. Hence $[Q/GL(N)]$ is a locally complete intersection stack of dimension 1 and irreducible. In particlar $[Q/GL(N)]$ is a reduced stack. □

Lemma 2.2.8. *Let E be a 0-semi-stable object with $v(E) = \varrho_X$. Then $\mathrm{Supp}(\pi_*(G^\vee \otimes E))$ is a point of Y.*

Proof. For E, we have a decomposition $E = \oplus_{i=1}^t E_i$ such that $\mathrm{Supp}(\pi_*(G^\vee \otimes E_i))$, $i = 1, ..., t$ are distinct t points of Y. We set $v(E_i) = (0, D_i, a_i)$. Since D_i are contained in the exceptional loci, $0 = \langle v(E)^2 \rangle = \sum_i (D_i^2)$ implies that $(D_i^2) = 0$ for all i. Thus we have $v(E_i) = a_i \varrho_X$ for all i, which implies that $\varrho_X = (\sum_i a_i)\varrho_X$. Since $\chi(G, E_i) > 0$, we have $a_i > 0$. Therefore $t = 1$. □

By Lemma 1.1.13, we get the following.

Lemma 2.2.9. (1) *$\mathbb{C}_x \in \mathcal{C}$ for all $x \in X$. In particular, we have a morphism $\varphi : X \to X^0$ by sending $x \in X$ to the S-equivalence class of $\mathbb{C}_x$.*
(2) *$\varphi(Z_i)$ is a point.*

If $\mathbb{C}_x$ is properly 0-semi-stable, then $\mathbb{C}_x$ is S-equivalent to $\oplus_j E_{ij}^{\oplus a'_{ij}}$ for an i.

Proposition 2.2.10. *There is an isomorphism $\psi : X^0 \to Y$ such that $\psi \circ \varphi : X \to Y$ coincides with π. In particular, X^0 is a normal projective surface.*

Proof. We keep the notation in the proof of Proposition 2.2.7. By Lemma 2.2.8, $\mathcal{F} := \pi_*(G^\vee \otimes \mathcal{E})$ is a flat family of coherent sheaves on Y such that $\mathrm{Supp}(\mathcal{F}_q)$ is a point for every $q \in Q$. Since the characteristic of the base field is zero, we have a morphism $Q \to S^r Y$, where $r = \mathrm{rk}\, G$ (cf. [F1], [F2]). Since the image is contained in the diagonal Y, we have a morphism $Q \to Y$. Hence we have a morphsim $\psi : X^0 \to Y$. By the construction of φ and ψ, $\pi = \psi \circ \varphi$. Since φ and ψ are projective birational morphisms between irreducible surfaces, φ and ψ are contractions. By using Lemma 2.2.9, we see that ψ is injective. Hence ψ is a finite morphism. Since Y is normal, ψ is an isomorphism. □

Lemma 2.2.11. *Assume that $\alpha \in \mathrm{NS}(X) \otimes \mathbb{Q}$ satisfies (2.12). Then K_{X^α} is the pull-back of a line bundle on X^0.*

Proof. Let $\mathcal{E}$ be the universal family on $X^\alpha \times X$. Let $p_S : S \times X \to S$ be the projection. Since X^α is smooth, the base change theorem implies that $\mathrm{Ext}^i_{p_{X^\alpha}}(\mathcal{E}, \mathcal{E})$, $i = 0, 1, 2$ are locally free sheaves on X^α and compatible with base changes. Since $\mathrm{Ext}^1_{p_{X^\alpha}}(\mathcal{E}, \mathcal{E})$ is the tangent bundle of X^α, we show that there is a symplectic form on $\mathrm{Ext}^1_{p_{X^\alpha}}(\mathcal{E}, \mathcal{E})$. For any point $y \in Y$, we take a very ample divisor D_2 on Y such that $y \notin D_2$, $|K_Y + D_2|$ contains a divisor D_1 with $y \notin D_1$. We set $U := Y \setminus (D_1 \cup D_2)$. Then U is an open neighborhood of y such that K_Y is trivial over U. Let $\widetilde{D}_i$ be the pull-back of D_i to X. Then we have $K_X = \mathcal{O}_X(\widetilde{D}_1 - \widetilde{D}_2)$. We set $V := \pi_\alpha^{-1}(\psi^{-1}(U))$. We shall prove that (i) the alternating pairing

$$\mathrm{Ext}^1_{p_V}(\mathcal{E}, \mathcal{E}) \times \mathrm{Ext}^1_{p_V}(\mathcal{E}, \mathcal{E}) \to \mathrm{Ext}^2_{p_V}(\mathcal{E}, \mathcal{E}) \tag{2.13}$$

is non-degenerate and (ii) $\mathrm{Ext}^2_{p_V}(\mathcal{E}, \mathcal{E}) \cong \mathcal{O}_V$. Since $\mathrm{Ext}^1_{p_{X^\alpha}}(\mathcal{E}, \mathcal{E})$ is the tangent bundle, this means that $K_V \cong \mathcal{O}_V$. Thus the claim holds.

We first note that there are isomorphisms

$$\mathrm{Ext}^i_{p_V}(\mathcal{E}, \mathcal{E}) \cong \mathrm{Ext}^i_{p_V}(\mathcal{E}, \mathcal{E}(\widetilde{D}_1)) \cong \mathrm{Ext}^i_{p_V}(\mathcal{E}, \mathcal{E}(\widetilde{D}_1 - \widetilde{D}_2)),\ i = 0, 1, 2, \tag{2.14}$$

which is compatible with the base change. By the Serre duality, the trace map $\mathrm{tr} : \mathrm{Ext}^2(\mathcal{E}_y, \mathcal{E}_y(K_X)) \to H^2(X, K_X)$ is an isomorphism for $y \in V$. Hence (ii) holds. By the Serre duality, the pairing $\mathrm{Ext}^1(\mathcal{E}_y, \mathcal{E}_y) \times \mathrm{Ext}^1(\mathcal{E}_y, \mathcal{E}_y(K_X)) \to \mathrm{Ext}^2(\mathcal{E}_y, \mathcal{E}_y(K_X)) \cong H^2(X, K_X)$ is non-degenerate. Combining this with (2.14), we get (i). □

Lemma 2.2.12. *Assume that $\alpha = 0$.*

1. *Assume that $p_i \in Y$ corresponds to $\bigoplus_{j=0}^{s_i} E_{ij}^{\oplus a_{ij}}$ via ψ, where E_{ij} are 0-stable objects. Then $\mathbb{C}_x, x \in Z_i$ are S-equivalent to $\bigoplus_{j=0}^{s_i} E_{ij}^{\oplus a_{ij}}$.*
2. *Let $E \in \mathcal{C}$ be a 0-twisted stable object. Then E is one of the following:*

$$\mathbb{C}_x\ (x \in X \setminus Z), \quad E_{ij},\ (1 \le i \le n, 0 \le j \le s_i). \tag{2.15}$$

3. *Every 0-dimensional object is generated by (2.15).*

Proof. By Proposition 2.2.10, (1) holds. We shall apply Lemma 2.2.5 to $\mathcal{E} = \mathcal{O}_\Delta \in \mathbf{D}(X \times X)$. Then (2) is a consequence of (1). It also follows from Lemma 1.1.13 (3). (3) follows from (2). □

Remark 2.2.13. If $\mathbf{b} = \mathbf{b}_0$, then $\pi_*(\mathcal{E})$ is a flat family of coherent sheaves on Y such that $\pi(\mathcal{E})_q$ is a point sheaf. Then we have a morphism $Q \to Y$. Thus we do not need the reducedness of Q in this case.

Definition 2.2.14. We set $Z_i^\alpha := \pi_\alpha^{-1}(\bigoplus_j E_{ij}^{\oplus a_{ij}}) = \pi_\alpha^{-1} \circ \psi^{-1}(p_i)$ and $Z^\alpha := \cup_i Z_i^\alpha$.

Lemma 2.2.15. *(cf.* [O-Y, Lem. 2.4]*) Let E_{ij} be 0-stable objects in Lemma 2.2.12. Assume that $-(\alpha, c_1(E_{ij})) > 0$ for all $j > 0$. Let F be a 0-semi-stable object such that $v(F) = v(E_{i0} \oplus \bigoplus_{j>0} E_{ij}^{\oplus b_j})$, $0 \le b_j \le a_{ij}$.*

1. *If $v(F) \ne \varrho_X$, then F is S-equivalent to $E_{i0} \oplus \bigoplus_{j>0} E_{ij}^{\oplus b_j}$ with respect to 0-stability.*
2. *Assume that F is S-equivalent to $E_{i0} \oplus \bigoplus_{j>0} E_{ij}^{\oplus b_j}$. Then the following conditions are equivalent.*
   - (a) *F is α-stable*
   - (b) *F is α-semi-stable*
   - (c) $\mathrm{Hom}(E_{ij}, F) = 0$ *for all $j > 0$.*
3. *Assume that F is α-stable. For a non-zero homomorphism $\phi : F \to E_{ij}$, $j > 0$, ϕ is surjective and $F' := \ker \phi$ is an α-stable object.*
4. *If there is a non-trivial extension*

$$0 \to F \to F'' \to E_{ij} \to 0 \tag{2.16}$$

and $b_k + \delta_{jk} \le a_{ik}$, then F'' is an α-stable object, where $\delta_{jk} = 0, 1$ according as $j \ne k$, $j = k$.

Proof. (1) Since $E := F \oplus \bigoplus_{j>0} E_{ij}^{\oplus (a_{ij} - b_j)}$ is a 0-semi-stable object with $v(E) = \varrho_X$ and $\mathrm{Supp}(\pi_*(G^\vee \otimes E)) = \mathrm{Supp}(\pi_*(G^\vee \otimes F)) \cup \{p_i\}$, Lemma 2.2.8 and Proposition 2.2.10 imply that the S-equivalence class of E corresponds to $p_i \in Y$. Hence E is S-equivalent to $\bigoplus_{j \ge 0} E_{ij}^{\oplus a_{ij}}$, which implies that F is S-equivalent to $E_{i0} \oplus \bigoplus_{j>0} E_{ij}^{\oplus b_j}$.

(2) It is sufficient to prove that (c) implies (a). Let $\psi : F \to I$ be a quotient of F. Since I and $\ker \psi$ are 0-dimensional objects, they are 0-semi-stable. Since $\mathrm{Hom}(E_{ij}, \ker \psi) = 0$ for $j > 0$, (1) implies that E_{i0} is a subobject of $\ker \psi$. Hence $v(I) = \sum_{j>0} b'_j v_{ij}$, which implies that F is α-stable.

(3) Since E_{ij} is irreducible, ϕ is surjective. By (1), $\ker \phi$ also satisfies the assumption of (2). Let $\psi : \ker \phi \to I$ be a quotient object. Since $\mathrm{Hom}(E_{ik}, F) = 0$ for $k > 0$, (2) implies that $\ker \phi$ is α-stable.

(4) Since $v(F) \neq \varrho_X$, (1) implies that F'' satisfies the assumption of (2). If $\operatorname{Hom}(E_{ik}, F'') \neq 0$, then $\operatorname{Hom}(E_{ik}, F) = 0$ implies that $k = j$ and we have a splitting of the exact sequence. Hence $\operatorname{Hom}(E_{ik}, F'') = 0$ for $k > 0$. Then (2) implies the claim. ☐

Corollary 2.2.16. *Assume that $-(\alpha, c_1(E_{ij})) > 0$ for all $j > 0$. We set $v := v(E_{i0} \oplus \oplus_{j>0} E_{ij}^{\oplus b_j})$, $0 \leq b_j \leq a_{ij}$ with $\langle v^2 \rangle = -2$.*

(1) $\dim \operatorname{Hom}(E, E_{ij}) = \max\{-\langle v, v(E_{ij})\rangle, 0\}$.

(2) *If $-\langle v, v(E_{ij})\rangle > 0$, then $M_{\mathcal{O}_X(1)}^{G,\alpha}(v) \cong M_{\mathcal{O}_X(1)}^{G,\alpha}(w)$, where $w = v + \langle v, v(E_{ij})\rangle v(E_{ij})$.*

Proof. (1) For $E \in M_{\mathcal{O}_X(1)}^{G,\alpha}(v)$, we set $n := \dim \operatorname{Hom}(E, E_{ij})$. Then we have a surjective morphism $\phi : E \to E_{ij}^{\oplus n}$. Then $F := \ker \phi$ is α-stable. Since $-2 \leq \langle v(F)^2 \rangle = \langle v(E)^2 \rangle - 2n(n + \langle v, v(E_{ij})\rangle)$, $n = -\langle v, v(E_{ij})\rangle$ or $n = 0$.

(2) If $-\langle v, v(E_{ij})\rangle > 0$, then $\dim \operatorname{Hom}(E, E_{ij}) = -\langle v, v(E_{ij})\rangle$, $\operatorname{Ext}^p(E, E_{ij}) = 0$, $p > 0$, and we have a morphism $\sigma : M_{\mathcal{O}_X(1)}^{G,\alpha}(v) \to M_{\mathcal{O}_X(1)}^{G,\alpha}(w)$. Conversely for $F \in M_{\mathcal{O}_X(1)}^{G,\alpha}(w)$, $\langle v(F), v(E_{ij})\rangle = -\langle v, v(E_{ij})\rangle > 0$. Hence $\operatorname{Hom}(F, E_{ij}) = 0$, which implies that $\dim \operatorname{Ext}^1(E_{ij}, F) = \langle v(F), v(E_{ij})\rangle$ and the universal extension gives an α-stable object E with $v(E) = v$. Therefore we also have the inverse of σ. ☐

We come to the main result of this subsection.

Theorem 2.2.17. *(cf.* [O-Y, Thm. 0.1]*)*

(1) *$X^0 \cong Y$ and the singular points $p_1, p_2, \dots, p_n$ of X^0 correspond to the S-equivalence classes of properly 0-twisted semi-stable objects.*

(2) *Assume that α satisfies that $(\alpha, D) \neq 0$ for all $D \in \operatorname{NS}(X)$ with $(D^2) = -2$ and $(c_1(\mathcal{O}_X(1)), D) = 0$. Then $X^\alpha = M_{\mathcal{O}_X(1)}^{G,\alpha}(\varrho_X)$. In particular $\pi_\alpha : X^\alpha \to X^0$ is the minimal resolution of the singularities.*

(3) *Let $\oplus_{j=0}^{s_i} E_{ij}^{\oplus a_{ij}}$ be the S-equivalence class corresponding to p_i. Then the matrix $(-\langle v(E_{ij}), v(E_{ik})\rangle)_{j,k \geq 0}$ is of affine type $\tilde{A}, \tilde{D}, \tilde{E}$. Assume that $a_{i0} = 1$. Then the singularity of X^0 at p_i is a rational double point of type A, D, E according as the type of the matrix $(-\langle v(E_{ij}), v(E_{ik})\rangle)_{j,k \geq 1}$.*

Proof. (1) By Proposition 2.2.10, $X^0 \cong Y$. Since $\varphi : X \to X^0$ is surjective, $y \in Y$ corresponds to the S-equivalence class of $\mathbb{C}_x$, $x \in \pi^{-1}(y)$. By Lemma 2.2.9, $\mathbb{C}_x$, $x \in \pi^{-1}(p_i)$ is not irreducible. Hence p_i corresponds to a properly 0-semi-stable objects. For a smooth point $y \in Y$, $\mathbb{C}_x$, $x \in \pi^{-1}(y)$ is irreducible. Therefore the second claim also holds. (2) is a consequence of Proposition 2.2.2 and Lemma 2.2.11.

(3) We note that

$$
\begin{aligned}
&\langle \varrho_X, v(E_{ij})\rangle = 0, \\
&\langle v(E_{ij}), v(E_{ij})\rangle = -2, \\
&\langle v(E_{ij}), v(E_{kl})\rangle \geq 0, \ (E_{ij} \neq E_{kl}).
\end{aligned}
\tag{2.17}
$$

As we see in Example 6.1.2 in appendix, we can apply Lemma 6.1.1 (1) to our situation. Hence the matrix $(-\langle v(E_{ij}), v(E_{ik})\rangle)_{j,k \geq 0}$ is of affine type $\tilde{A}, \tilde{D}, \tilde{E}$. Then we may assume that $a_{i0} = 1$ for all i. By Lemma 6.1.1 (2), we can choose an α with $-\langle v(E_{ij}), \alpha\rangle > 0$ for all $j > 0$. Let $\mathcal{E}^\alpha$ be the universal family on $X \times X^\alpha$. (3) is a consequence of the following lemma. ☐

Lemma 2.2.18. *Assume that α satisfies $-\langle v(E_{ij}), \alpha\rangle > 0$ for all $j > 0$.*

(1) *We set*

$$C_{ij}^\alpha := \{x^\alpha \in X^\alpha \mid \operatorname{Hom}(\mathcal{E}_{|X \times \{x^\alpha\}}, E_{ij}) \neq 0\}, j > 0. \tag{2.18}$$

Then C_{ij}^α is a smooth rational curve.

(2)

$$Z_i^\alpha = \{x^\alpha \in X^\alpha \mid \operatorname{Hom}(E_{i0}, \mathcal{E}_{|X \times \{x^\alpha\}}) \neq 0\} = \cup_j C_{ij}^\alpha. \tag{2.19}$$

(3) *$\cup_j C_{ij}^\alpha$ is simple normal crossing and $(C_{ij}^\alpha, C_{ik}^\alpha) = \langle v(E_{ij}), v(E_{ik})\rangle$.*

Proof. (1) By our choice of α, $\operatorname{Hom}(E_{ij}, \mathcal{E}_{|X \times \{x^\alpha\}}) = 0$ for all $x^\alpha \in X^\alpha$. If $C_{ij}^\alpha = \emptyset$, then $\chi(E_{ij}, \mathcal{E}_{|X \times \{x^\alpha\}}) = 0$ implies that $\operatorname{Hom}(\mathcal{E}_{|X \times \{x^\alpha\}}, E_{ij}) = \operatorname{Ext}^1(\mathcal{E}_{|X \times \{x^\alpha\}}, E_{ij}) = 0$. Then $\Phi_{X \to X^\alpha}^{\mathcal{E}^\vee}(E_{ij}) = 0$, which is a contradiction. Therefore $C_{ij}^\alpha \neq \emptyset$. In order to prove the smoothness, we consider the moduli space of coherent systems

$$N(\varrho_X, v(E_{ij})) := \{(E, V) \mid E \in X^\alpha, V \subset \operatorname{Hom}(E, E_{ij}), \dim_{\mathbb{C}} V = 1\}. \tag{2.20}$$

We have a natural projection $\iota : N(\varrho_X, v(E_{ij})) \to X^\alpha$ whose image is C_{ij}^α. For $(E, V) \in N(\varrho_X, v(E_{ij}))$, we have a homomorphism $\xi : E \to E_{ij} \otimes V^\vee$. The Zariski tangent space at (E, V) is $\operatorname{Hom}(E, E \to E_{ij} \otimes V^\vee)$. By

Lemma 2.2.15 (3), ξ is surjective and $\ker \xi \in M^{G,\alpha}_{\mathcal{O}_X(1)}(\varrho_X - v(E_{ij}))$. In particular $\operatorname{Hom}(E, E \to E_{ij} \otimes V^\vee) \cong \operatorname{Ext}^1(E, \ker \xi)$. Conversely for $F \in M^{G,\alpha}_{\mathcal{O}_X(1)}(\varrho_X - v(E_{ij}))$ and a non-trivial extension

$$0 \to F \to E \to E_{ij} \to 0, \tag{2.21}$$

Lemma 2.2.15 (4) implies that $E \in X^\alpha$ and $E \to E_{ij}$ defines an element of $N(\varrho_X, v(E_{ij}))$. By Corollary 2.2.16 (1) and our choice of α, $\operatorname{Hom}(F, E_{ij}) = \operatorname{Hom}(E_{ij}, F) = 0$. Hence $\dim \operatorname{Ext}^1(E_{ij}, F) = 2$. Since $M^{G,\alpha}_{\mathcal{O}_X(1)}(\varrho_X - v(E_{ij}))$ is a reduced one point, we see that $N(\varrho_X, v(E_{ij}))$ is isomorphic to $\mathbb{P}^1$. We show that $\iota : N(\varrho_X, v(E_{ij})) \to X^\alpha$ is a closed immersion. For $(E, V) \in N(\varrho_X, v(E_{ij}))$, $\dim \operatorname{Hom}(E, E_{ij}) = \dim \operatorname{Hom}(\ker \xi, E_{ij}) + 1 = 1$. Hence ι is injective. We also see that $\iota_* : \operatorname{Ext}^1(E, \ker \xi) \to \operatorname{Ext}^1(E, E)$ is injective. Therefore ι is a closed immersion.

(2) By our choice of α, $\operatorname{Hom}(E_{i0}, \mathcal{E}_{|X \times \{x^\alpha\}}) \neq 0$ for $x^\alpha \in Z_i^\alpha$. Conversely if $\operatorname{Hom}(E_{i0}, \mathcal{E}_{|X \times \{x^\alpha\}}) \neq 0$, then Lemma 2.2.8 implies that $\operatorname{Supp}(\pi_*(G^\vee \otimes \mathcal{E}_{|X \times \{x^\alpha\}})) = \{p_i\}$. Since $\operatorname{Supp}(\pi_*(G^\vee \otimes \mathcal{E}_{|X \times \{x^\alpha\}}))$ depends only on the S-equivalence class of $\mathcal{E}_{|X \times \{x^\alpha\}}$, we have $\psi(\pi_\alpha(x^\alpha)) = p_i$. Thus $x^\alpha \in Z_i^\alpha$. Therefore we have the first equality. By the choice of α, we also get $Z_i^\alpha \subset \cup_j C_{ij}^\alpha$. If $\operatorname{Hom}(\mathcal{E}_{|X \times \{x^\alpha\}}, E_{ij}) \neq 0$, $j > 0$, then we see that $\operatorname{Supp}(\pi_*(G^\vee \otimes \mathcal{E}_{|X \times \{x^\alpha\}})) = \{p_i\}$, which implies that $x^\alpha \in Z_i^\alpha$. Thus the second claim also holds.

(3) Since $(-\langle v(E_{ij}), v(E_{ik}) \rangle)_{j,k \geq 1}$ is of ADE-type, by using Corollary 2.2.16, we can show that $M^{G,\alpha}_{\mathcal{O}_X(1)}(v) \cong M^{G,\alpha}_{\mathcal{O}_X(1)}(v(E_{i0}))$ for $v = v(E_{i0} \oplus \oplus_{j>0} E_{ij}^{\oplus b_j})$, $0 \leq b_j \leq a_{ij}$ with $\langle v^2 \rangle = -2$. In particular, they are non-empty. Then by a similar arguments in [O-Y, Prop. 2.9], we can also show that $\cup_j C_{ij}^\alpha$ is simple normal crossing and $(C_{ij}^\alpha, C_{ik}^\alpha) = \langle v(E_{ij}), v(E_{ik}) \rangle$. For another proof, see Corollary 2.3.12. □

2.3. **Fourier-Mukai transforms on X.** We keep the notations in subsection 2.2. Assume that X^α consists of α-stable objects. Let $\mathcal{E}^\alpha$ be a universal family on $X \times X^\alpha$. We have an equivalence $\Phi^{(\mathcal{E}^\alpha)^\vee}_{X \to X^\alpha} : \mathbf{D}(X) \to \mathbf{D}(X^\alpha)$. If $\mathcal{F}^\alpha$ be another universal family, then we see that

$$\Phi^{(\mathcal{E}^\alpha)^\vee}_{X \to X^\alpha} \circ \Phi^{\mathcal{F}^\alpha}_{X^\alpha \to X} = \Phi^{\mathcal{O}_\Delta(L)}_{X^\alpha \to X^\alpha}[-2], L \in \operatorname{Pic}(X^\alpha). \tag{2.22}$$

Let Γ^α be the closure of the graph of the rational map $\pi_\alpha^{-1} \circ \pi$:

$$\begin{array}{ccc} \Gamma^\alpha & \longrightarrow & X^\alpha \\ \downarrow & & \downarrow{\scriptstyle \pi_\alpha} \\ X & \underset{\pi}{\longrightarrow} & Y. \end{array} \tag{2.23}$$

Lemma 2.3.1. (1) *We may assume that* $\mathcal{E}^\alpha_{|X \times (X^\alpha \setminus Z^\alpha)} \cong \mathcal{O}_{\Gamma^\alpha | X \times (X^\alpha \setminus Z^\alpha)}$.

(2) $\mathcal{E}^\alpha$ *is characterized by* $\mathcal{E}^\alpha_{|X \times (X^\alpha \setminus Z^\alpha)}$ *and* $\det \Phi^{(\mathcal{E}^\alpha)^\vee}_{X \to X^\alpha}(G)$.

Proof. (1) We note that $\mathcal{E}^\alpha_{|X \times (X^\alpha \setminus Z^\alpha)} \cong (\mathcal{O}_{\Gamma^\alpha} \otimes p_{X^\alpha}^*(L))_{|X \times (X^\alpha \setminus Z^\alpha)}$, where $L \in \operatorname{Pic}(X^\alpha \setminus Z^\alpha)$. We also denote an extension of L to X^α by L. Then $\mathcal{E}^\alpha \otimes p_{X^\alpha}^*(L^\vee)$ is a desired universal family.

(2) Assume that $\mathcal{E}^\alpha_{|X \times (X^\alpha \setminus Z^\alpha)} \cong (\mathcal{E}^\alpha \otimes p_{X^\alpha}^*(L))_{|X \times (X^\alpha \setminus Z^\alpha)}$ and $\det \Phi^{(\mathcal{E}^\alpha)^\vee}_{X \to X^\alpha}(G) \cong \det \Phi^{(\mathcal{E}^\alpha \otimes p_{X^\alpha}^*(L))^\vee}_{X \to X^\alpha}(G)$. Then $L_{|X^\alpha \setminus Z^\alpha} \cong \mathcal{O}_{X^\alpha \setminus Z^\alpha}$ and $L^{\otimes \operatorname{rk} G} \cong \mathcal{O}_{X^\alpha}$. In order to prove $L \cong \mathcal{O}_{X^\alpha}$, it is sufficient to prove the injectivity of the restriction map

$$r : \operatorname{Pic}(X^\alpha) \to \operatorname{Pic}(X^\alpha \setminus Z^\alpha) \times \prod_{i,j} \operatorname{Pic}(C_{ij}^\alpha). \tag{2.24}$$

If $L_{|X^\alpha \setminus Z^\alpha} \cong \mathcal{O}_{X^\alpha \setminus Z^\alpha}$, then we can write $L = \mathcal{O}_X(\sum_{i,j} r_{ij} C_{ij}^\alpha)$. Since the intersection matrix $((C_{ij}^\alpha, C_{ik}^\alpha))_{j,k}$ is negative definite, $\deg(L_{|C_{ij}^\alpha}) = \sum_k r_{ik}(C_{ik}^\alpha, C_{ij}^\alpha) = 0$ for all i, j implies that $r_{ij} = 0$ for all i, j. Thus r is injective. □

Definition 2.3.2. We set $\Lambda^\alpha := \Phi^{(\mathcal{E}^\alpha)^\vee}_{X \to X^\alpha}[2]$.

Lemma 2.3.3. $\mathcal{O}_X(n) \otimes \_$ *and* Λ^α *are commutative.*

Proof. Let D be an effective divisor on X such that $D \cap Z = \emptyset$. It is sufficient to prove that

$$\mathcal{E}^\alpha \otimes (\mathcal{O}_X(-D) \boxtimes \mathcal{O}_{X^\alpha}(D)) \cong \mathcal{E}^\alpha. \tag{2.25}$$

We note that $\mathcal{E}^\alpha \cong \mathcal{O}_{\Gamma^\alpha}$ over $X^\alpha \setminus Z^\alpha$. Obviously the claim holds over $X^\alpha \setminus Z^\alpha$. By Lemma 2.3.1, we shall show that $\det \Lambda^\alpha(G(D)) \cong \det(\Lambda^\alpha(G)(D))$. We have an exact triangle

$$(\mathcal{E}^\alpha)^\vee \to (\mathcal{E}^\alpha)^\vee(D) \to (\mathcal{E}^\alpha)^\vee_{|D}(D) \to (\mathcal{E}^\alpha)^\vee[1]. \tag{2.26}$$

Since $(\mathcal{E}^\alpha)^\vee_{|D}(D) \cong \mathcal{O}_{\Delta|D}(D)[-2]$, we have an exact triangle

$$\Lambda^\alpha(G) \to \Lambda^\alpha(G(D)) \to G_{|D}(D) \to \Lambda^\alpha(G)[1]. \tag{2.27}$$

Hence we get $\det \Lambda^\alpha(G(D)) \cong (\det \Lambda^\alpha(G))((\operatorname{rk} G)D) \cong \det(\Lambda^\alpha(G)(D))$. □

Proposition 2.3.4. (1) *$G^\alpha := \Lambda^\alpha(G)$ is a locally free sheaf and $\mathbf{R}\pi_{\alpha*}(G^{\alpha\vee} \otimes G^\alpha) = \pi_{\alpha*}(G^{\alpha\vee} \otimes G^\alpha)$.*

(2) *$\Lambda^\alpha(E_{ij})[j]$ is a sheaf, where $j = -1$ or 0 according as $(\alpha, c_1(E_{ij})) < 0$ or $(\alpha, c_1(E_{ij})) > 0$.*

(3) *We set $\mathcal{A}^\alpha := \pi_{\alpha*}(G^{\alpha\vee} \otimes G^\alpha)$. Then $\mathcal{A}^\alpha$ is a reflexive sheaf on Y. Under the identification $X^\alpha \setminus Z^\alpha \cong X \setminus Z$, $G^\alpha_{|X^\alpha \setminus Z^\alpha}$ corresponds to $G_{|X\setminus Z}$. Hence we have an isomorphism $\mathcal{A} \cong \mathcal{A}^\alpha$.*

(4) *We identify $\operatorname{Coh}_{\mathcal{A}}(Y)$ with $\operatorname{Coh}_{\mathcal{A}^\alpha}(Y)$ via $\mathcal{A} \cong \mathcal{A}^\alpha$. Then we have a commutative diagram*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\Lambda^\alpha} & \Lambda^\alpha(\mathcal{C}) \\ {\scriptstyle \mathbf{R}\pi_*\mathcal{H}om(G,\ )}\downarrow & & \downarrow{\scriptstyle \mathbf{R}\pi_{\alpha*}\mathcal{H}om(G^\alpha,\ )} \\ \operatorname{Coh}_{\mathcal{A}}(Y) & = & \operatorname{Coh}_{\mathcal{A}^\alpha}(Y) \end{array} \tag{2.28}$$

In particular G^α gives a local projective generator of $\Lambda^\alpha(\mathcal{C})$.

(5) *We set*

$$\begin{aligned} S^\alpha :=& \{\Lambda^\alpha(E_{ij})[-1] | i,j\} \cap \operatorname{Coh}(X^\alpha), \\ \mathcal{T}^\alpha :=& \{E \in \operatorname{Coh}(X^\alpha) | \operatorname{Hom}(E,c) = 0, c \in S^\alpha\}, \\ \mathcal{S}^\alpha :=& \{E \in \operatorname{Coh}(X^\alpha) |\ E \text{ is a successive extension of subsheaves of } c \in S^\alpha\ \}. \end{aligned} \tag{2.29}$$

Then $(\mathcal{T}^\alpha, \mathcal{S}^\alpha)$ is a torsion pair of $\operatorname{Coh}(X^\alpha)$ and $\Lambda^\alpha(\mathcal{C})$ is the tilting of $\operatorname{Coh}(X^\alpha)$ with respect to $(\mathcal{T}^\alpha, \mathcal{S}^\alpha)$.

(6) *Let G' be a local projective generator of $\mathcal{C}$. Then Λ^α induces an isomorphism $\mathcal{M}_H^{G'}(v)^{ss} \to \mathcal{M}_H^{\Lambda^\alpha(G')}(\Lambda^\alpha(v))^{ss}$.*

Proof. (1) We note that $\operatorname{Hom}(\mathcal{E}^\alpha_{|X\times\{x^\alpha\}}, G[i]) \cong \operatorname{Hom}(G, \mathcal{E}^\alpha_{|X\times\{x^\alpha\}}[2-i])^\vee = 0$ for $i \neq 2$ and $x^\alpha \in X^\alpha$. By the base change theorem, G^α is a locally free sheaf. By using Lemma 2.3.3 and the ampleness of $\mathcal{O}_Y(1)$, we have

$$\begin{aligned} H^0(Y, R^i\pi_{\alpha*}(G^{\alpha\vee} \otimes G^\alpha)(n)) =& \operatorname{Hom}(\Lambda^\alpha(G), \Lambda^\alpha(G)(n)[i]) \\ =& \operatorname{Hom}(\Lambda^\alpha(G), \Lambda^\alpha(G(n))[i]) \\ =& \operatorname{Hom}(G, G(n)[i]) = H^0(Y, R^i\pi_*(G^\vee \otimes G)(n)) = 0 \end{aligned} \tag{2.30}$$

for $n \gg 0$ and $i \neq 0$. Therefore $R^i\pi_*(G^{\alpha\vee} \otimes G^\alpha) = 0$, $i \neq 0$ and the claim holds.

(2) If $(\alpha, c_1(E_{ij})) < 0$, then $\operatorname{Hom}(\mathcal{E}^\alpha_{|X\times\{x^\alpha\}}, E_{ij}[2]) \cong \operatorname{Hom}(E_{ij}, \mathcal{E}^\alpha_{|X\times\{x^\alpha\}})^\vee = 0$ for $x^\alpha \in X^\alpha$. Since $\operatorname{Hom}(\mathcal{E}^\alpha_{X\times|\{x^\alpha\}}, E_{ij}) = 0$ if $x^\alpha \notin Z_i^\alpha$, we see that $\Lambda^\alpha(E_{ij})[-1]$ is a torsion sheaf whose support is contained in Z_i^α.

If $(\alpha, c_1(E_{ij})) > 0$, then $\operatorname{Hom}(\mathcal{E}^\alpha_{|X\times\{x^\alpha\}}, E_{ij}) = 0$ for $x^\alpha \in X^\alpha$. Since $\operatorname{Hom}(\mathcal{E}^\alpha_{|X\times\{x^\alpha\}}, E_{ij}[2]) = 0$ if $x^\alpha \notin Z_i^\alpha$, we see that $\Lambda^\alpha(E_{ij})$ is a torsion sheaf whose support is contained in Z_i^α.

(3) By the claim (1) and [E, Lem. 2.1], $\mathcal{A}^\alpha$ is a reflexive sheaf. Since $\mathcal{E}^\alpha$ is isomorphic to $\mathcal{O}_{\Gamma^\alpha}$ over $X^\alpha \setminus Z^\alpha$, we get $\Lambda^\alpha(G)_{|X^\alpha\setminus Z^\alpha} \cong \pi_\alpha^{-1} \circ \pi(G_{|X\setminus Z})$. Hence the second claim also follows.

(4) For $E \in \mathcal{C}$, we first prove that $\mathbf{R}\pi_*(G^{\alpha\vee} \otimes \Lambda^\alpha(E)) \in \operatorname{Coh}_{\mathcal{A}^\alpha}(Y)$. As in the proof of (1), we have

$$\begin{aligned} H^i(Y, \mathbf{R}\pi_*(G^{\alpha\vee} \otimes \Lambda^\alpha(E))(n)) =& \operatorname{Hom}(G^\alpha, \Lambda^\alpha(E)(n)[i]) \\ =& \operatorname{Hom}(G, E(n)[i]) = 0 \end{aligned} \tag{2.31}$$

for $i \neq 0$, $n \gg 0$. Therefore $H^i(\mathbf{R}\pi_*(G^{\alpha\vee} \otimes \Lambda^\alpha(E))) = 0$ for $i \neq 0$. For $E \in \mathcal{C}$, we take an exact sequence

$$G(-m)^{\oplus M} \to G(-n)^{\oplus N} \to E \to 0 \tag{2.32}$$

Then we have a diagram

$$\begin{array}{ccccccc} \mathcal{A}(-m)^{\oplus M} & \longrightarrow & \mathcal{A}(-n)^{\oplus N} & \longrightarrow & \pi_*(G^\vee \otimes E) & \longrightarrow & 0 \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\psi} & & & & \\ \mathcal{A}^\alpha(-m)^{\oplus M} & \longrightarrow & \mathcal{A}^\alpha(-n)^{\oplus N} & \longrightarrow & \pi_*(G^{\alpha\vee} \otimes \Lambda^\alpha(E)) & \longrightarrow & 0 \end{array} \tag{2.33}$$

which is commutative over $Y^* := Y \setminus \{p_1, p_2, \dots, p_n\}$, where ϕ and ψ are the isomorphisms induced by $\mathcal{A} \cong \mathcal{A}^\alpha$. Let $j : Y^* \hookrightarrow Y$ be the inclusion. Since $\mathcal{H}om(\mathcal{A}, \mathcal{A}^\alpha) \to j_*j^*\mathcal{H}om(\mathcal{A}, \mathcal{A}^\alpha)$ is an isomorphism, (2.33) is commutative, which induces an isomorphism $\xi : \pi_*(G^\vee \otimes E) \to \pi_*(G^{\alpha\vee} \otimes \Lambda^\alpha(E))$. It is easy to see that the construction of ξ is functorial and defines an isomorphism $\mathbf{R}\pi_*\mathcal{H}om(G,\ ) \cong \mathbf{R}\pi_*\mathcal{H}om(G^\alpha,\ ) \circ \Lambda^\alpha$.

(5) Since Λ^α is an equivalence, $\Lambda^\alpha(E_{ij})$ are irreducible objects of $\Lambda^\alpha(\mathcal{C})$. By Lemma 1.1.7 and Proposition 1.1.19, we get the claim.

(6) We note that the proof of (1) implies that $\Lambda^\alpha(G')$ is a local projective generator of $\Lambda^\alpha(\mathcal{C})$. By Lemma 2.3.3, $\chi(G', E(n)) = \chi(\Lambda^\alpha(G'), \Lambda^\alpha(E)(n))$. Hence the claim holds. □

Remark 2.3.5. If $\mathcal{C} = {}^{-1}\operatorname{Per}(X/Y)$, then $\mathcal{O}_X \in {}^{-1}\operatorname{Per}(X/Y)$ and $\Lambda^\alpha(\mathcal{O}_X)$ is a line bundle on X^α. Hence we may assume that $\Lambda^\alpha(\mathcal{O}_X) \cong \mathcal{O}_{X^\alpha}$. Then $\operatorname{Hom}(\mathcal{O}_{X^\alpha}, \Lambda^\alpha(\mathcal{O}_{C_{ij}}(-1))[n]) = 0$ for all n. Thus $\Lambda^\alpha(\mathcal{O}_{C_{ij}}(-1))[n]$ is a successive extensions of $\mathcal{O}_{C_{ik}}(-1)$. We also get $\operatorname{Hom}(\mathcal{O}_{X^\alpha}, \Lambda^\alpha(\mathcal{O}_{Z_i})) = \mathbb{C}$ and $\operatorname{Hom}(\mathcal{O}_{X^\alpha}, \Lambda^\alpha(\mathcal{O}_{Z_i})[n]) = 0$ for $n \neq 0$.

Since Λ^α is an equivalence with $\Lambda^\alpha(\varrho_X) = \varrho_{X^\alpha}$, we have the following corollary.

Corollary 2.3.6. *For a general α, the equivalence*

$$\Lambda^\alpha : \mathcal{C} \to \Lambda^\alpha(\mathcal{C})$$

induces an isomorphism:

$$\Lambda^\alpha : \mathcal{M}_{\mathcal{O}_X(1)}^{G,\beta}(\varrho_X)^{ss} \to \mathcal{M}_{\mathcal{O}_{X^\alpha}(1)}^{G^\alpha,\Lambda^\alpha(\beta)}(\varrho_{X^\alpha})^{ss},$$

where $\beta \in \varrho_X^\perp$.

2.3.1. *Wall and chambers.* For the 0-stable objects E_{ij} in Theorem 2.2.17, we set $v_{ij} := v(E_{ij})$. By Lemma 2.2.5, $\{E_{ij}\}$ is the set of irreducible objects E with $\operatorname{Supp}(E) \subset \cup_i Z_i$. Let $\mathfrak{g}_i$ be the finite Lie algebra whose Cartan matrix is $(-\langle v_{ij}, v_{ik}\rangle_{j,k\geq 1})$ and

$$R_i := \left\{ u = \sum_{j>0} n'_{ij} v_{ij} \,\middle|\, \langle u^2 \rangle = -2, n'_{ij} \geq 0 \right\}. \tag{2.34}$$

Then R_i is identified with the set of positive roots of $\mathfrak{g}_i$. In particular, R_i is a finite set.

Definition 2.3.7. For $u \in \cup_i R_i$, we define the wall as

$$W_u := \left\{ \alpha \in \operatorname{NS}(X) \otimes \mathbb{R} \,\middle|\, \frac{\langle u, \alpha\rangle}{\langle u, v(G)\rangle} = \frac{\langle v, \alpha\rangle}{\langle v, v(G)\rangle} \right\}. \tag{2.35}$$

A connected component of $\operatorname{NS}(X) \otimes \mathbb{R} \setminus \cup_u W_u$ is called a chamber.

Remark 2.3.8. If $v = \varrho_X$, then $W_u = u^\perp$.

Lemma 2.3.9. *Let v be the Mukai vector of a 0-dimensional object E, which is primitive.*

(1) *$\overline{M}_{\mathcal{O}_X(1)}^{G,\alpha}(v)$ consists of α-twisted stable objects if and only if $\alpha \notin \cup_u W_u$. We say that α is general with respect to v.*
(2) *If α is general with respect to v, then the virtual Hodge number of $M_{\mathcal{O}_X(1)}^{G,\alpha}(v)$ does not depend on the choice of α. In particular, the non-emptyness of $M_{\mathcal{O}_X(1)}^{G,\alpha}(v)$ does not depend on the choice of α.*

Proof. (1) For $E \in \overline{M}_{\mathcal{O}_X(1)}^{G,\alpha}(v)$, we assume that E is S-equivalent to $\oplus_{i=1}^n E_i$. If $\langle v(E_i)^2\rangle = 0$ for all i, then $v(E_i) \in \mathbb{Z}_{>0}\varrho_X$. Hence $v = \sum_{i=1}^n v(E_i)$ is not primitive. Therefore we may assume that $\langle v(E_1)^2\rangle = -2$. By the α-stability of E_1, $\operatorname{Supp}(E_1) \subset Z_i$ for an i. Since E_1 is generated by $\{E_{ij} | 0 \leq j \leq s_i\}$, $v(E_1) \in \oplus_{j=0}^{s_i} \mathbb{Z}_{\geq 0} v_{ij}$. Then we see that $v(E_1) \in \pm R_i + \mathbb{Z}\varrho_X$. Therefore the claim holds. (2) The proof is similar to that of [Y3, Prop. 2.6]. □

Lemma 2.3.10. (1) *Let $w_1 := v_{i0} + \sum_{j=1}^{s_i} n_{ij} v_{ij}$, $n_{ij} \geq 0$ be a Mukai vector with $\langle w_1^2\rangle \geq -2$. Then there is an α-twisted stable object E with $v(E) = w_1$ for a general α.*
(2) *Let $w_2 \in R_i$ be a non-zero Mukai vector. Then there is an α-twisted stable object E with $v(E) = w_2$ for a general α.*

Proof. (1) By Proposition 2.3.16 below, we may assume that $\mathcal{C} = \operatorname{Per}(X'/Y, \mathbf{b}_1, ..., \mathbf{b}_n)$. The claim follows from Lemme 2.3.19 below and Lemma 2.3.9 (2). Instead of using Lemma 2.3.19, we can also use Corollary 2.2.16 to show the claim for a special α.

(2) We set $w_1 := \sum_{j=0}^{s_i} a_{ij} v_{ij} - w_2$. Then w_1 is the Mukai vector in (1). We can take a general element $\alpha \in \operatorname{NS}(X) \otimes \mathbb{Q}$ such that $\langle \alpha, w_1\rangle = 0$. Then α is general with respect to w_1 and we have a α-twisted stable object E with $v(E) = w_1$. We consider $X^{\alpha'}$ such that α' is sufficiently close to α and $\langle \alpha', v(E)\rangle > 0$. Since $\Lambda^{\alpha'}$ is an equivalence, there is a morphism $\phi : E \to \mathcal{E}^{\alpha'}_{|\{y\}\times X}$, where $y \in X^{\alpha'}$. By our choice of α, $\operatorname{coker}\phi$ is an α-twisted stable object with $v(\operatorname{coker}\phi) = w_2$. Then the claim follows from Lemma 2.3.9 (2). □

2.3.2. *A special chamber.* We take $\alpha \in \varrho_X^\perp$ with $-\langle v(E_{ij}), \alpha\rangle > 0$, $j > 0$.

Lemma 2.3.11. *$\Lambda^\alpha(E_{ij})[-1]$, $j > 0$ is a line bundle on C_{ij}^α. We set $\Lambda^\alpha(E_{ij}) := \mathcal{O}_{C_{ij}^\alpha}(b_{ij}^\alpha)[1]$.*

Proof. We note that $\Lambda^\alpha(E_{ij}) \overset{\mathbf{L}}{\otimes} \mathbb{C}_{x^\alpha} = \mathbf{R}\operatorname{Hom}(\mathcal{E}^\alpha_{|X\times\{x^\alpha\}}, E_{ij}[2])$. Then $H^k(\Lambda(E_{ij}) \overset{\mathbf{L}}{\otimes} \mathbb{C}_{x^\alpha}) = 0$ for $k \neq -1, -2$. Hence $H^k(\Lambda^\alpha(E_{ij})) = 0$ for $k \neq -1, -2$ and $H^{-2}(\Lambda^\alpha(E_{ij}))$ is a locally free sheaf. By the proof of Theorem 2.2.17 (3), $\operatorname{Supp}(H^k(\Lambda^\alpha(E_{ij}))) \subset C^\alpha_{ij}$ for all k. Hence $H^{-2}(\Lambda^\alpha(E_{ij})) = 0$, which implies that $\Lambda^\alpha(E_{ij})[-1] \in \operatorname{Coh}(X^\alpha)$. Since $\operatorname{Hom}(\mathbb{C}_{x^\alpha}, \Lambda^\alpha(E_{ij})[-1]) = \operatorname{Hom}(\mathcal{E}^\alpha_{|X\times\{x^\alpha\}}, E_{ij}[-1]) = 0$, $\Lambda^\alpha(E_{ij})[-1]$ is purely 1-dimensional. We set $C := \operatorname{Div}(\Lambda^\alpha(E_{ij})[-1])$. Then $(C^2) = \langle v(\Lambda^\alpha(E_{ij})[-1])^2\rangle = \langle v(E_{ij})^2\rangle = -2$, which implies that $C = C^\alpha_{ij}$. Therefore $\Lambda^\alpha(E_{ij})[-1]$ is a line bundle on C^α_{ij}. ☐

Corollary 2.3.12. (1) $(C^\alpha_{ij}, C^\alpha_{i'j'}) = \langle v(E_{ij}), v(E_{i'j'})\rangle$.
(2) *$\{C^\alpha_{ij}\}$ is a simple normal crossing divisor.*

Proof. (1) By Lemma 2.3.11, $(C^\alpha_{ij}, C^\alpha_{i'j'}) = \langle v(\Lambda^\alpha(E_{ij})), v(\Lambda^\alpha(E_{i'j'}))\rangle = \langle v(E_{ij}), v(E_{i'j'})\rangle$. Then (2) also follows. ☐

E_{i0} is a subobject of $\mathcal{E}_{|X\times\{x^\alpha\}}$ for $x^\alpha \in Z^\alpha_i$ and we have an exact sequence

$$0 \to E_{i0} \to \mathcal{E}_{|X\times\{x^\alpha\}} \to F \to 0,\ x^\alpha \in Z^\alpha_i \tag{2.36}$$

where F is a 0-semi-stable object with $\operatorname{gr}(F) = \oplus_{j=1}^{s_i} E_{ij}^{\oplus a_{ij}}$. Then we get an exact sequence

$$0 \to \Lambda^\alpha(F)[-1] \to \Lambda^\alpha(E_{i0}) \to \mathbb{C}_{x^\alpha} \to 0 \tag{2.37}$$

in $\operatorname{Coh}(X^\alpha)$. Thus $\Lambda^\alpha(E_{i0}) \in \operatorname{Coh}(X^\alpha)$.

Definition 2.3.13. We set $A^\alpha_{i0} := \Lambda^\alpha(E_{i0})$ and $A^\alpha_{ij} := \Lambda^\alpha(E_{ij}) = \mathcal{O}_{C^\alpha_{ij}}(b^\alpha_{ij})[1]$ for $j > 0$.

Lemma 2.3.14. (1) $\operatorname{Hom}(A^\alpha_{i0}, A^\alpha_{ij}[-1]) = \operatorname{Ext}^1(A^\alpha_{i0}, A^\alpha_{ij}[-1]) = 0$.
(2) *We set $\mathbf{b}^\alpha_i := (b^\alpha_{i1}, b^\alpha_{i2}, \dots, b^\alpha_{is_i})$. Then $A^\alpha_{i0} \cong A_0(\mathbf{b}^\alpha_i)$. In particular, $\operatorname{Hom}(A^\alpha_{i0}, \mathbb{C}_{x^\alpha}) = \mathbb{C}$ for $x^\alpha \in Z^\alpha_i$.*

Proof. (1) We have

$$\begin{aligned}\operatorname{Hom}(A^\alpha_{i0}, A^\alpha_{ij}[k]) &= \operatorname{Hom}(\Lambda^\alpha(E_{i0}), \Lambda^\alpha(E_{ij})[k])\\ &= \operatorname{Hom}(E_{i0}, E_{ij}[k]) = 0\end{aligned} \tag{2.38}$$

for $k = -1, 0$.

(2) By (2.37) and (1), we can apply Lemma 1.2.18 and get $A^\alpha_{i0} = A_0(\mathbf{b}^\alpha_i) = A_{p_i}$. ☐

Remark 2.3.15. Assume that $\alpha \in v_0^\perp$ satisfies $-\langle v(E_{ij}), \alpha\rangle < 0$, $j > 0$. Then $\Phi(E_{ij})[2] = \mathcal{O}_{C^\alpha_{ij}}(b''_{ij})$, $j > 0$ and $\Phi(E_{i0})[2] = A_0(\mathbf{b}''_i)[1]$ belong to $\operatorname{Per}(X^\alpha/Y, \mathbf{b}''_1, ..., \mathbf{b}''_n)^*$, where $\mathbf{b}''_i := (b''_{i1}, ..., b''_{is_i})$.

By Proposition 2.3.4, we have the following result.

Proposition 2.3.16. *If $-\langle \alpha, v(E_{ij})\rangle > 0$ for all $j > 0$, then Λ^α induces an equivalence*

$$\mathcal{C} \to \operatorname{Per}(X^\alpha/Y, \mathbf{b}^\alpha_1, ..., \mathbf{b}^\alpha_n),$$

where $\mathbf{b}^\alpha_i = (b^\alpha_{i1}, ..., b^\alpha_{is_i})$.

Proposition 2.3.17. *Assume that there is a $\beta \in \varrho_X^\perp$ such that $\mathbb{C}_x$ are β-stable for all $x \in X$.*
(1) *We set $\mathcal{F} := \mathcal{E}^{\alpha\vee}[2]$. Then we have an isomorphism*

$$\begin{array}{rcl} X & \to & M^{G^\alpha,\Lambda^\alpha(\beta)}_{\mathcal{O}_{X^\alpha}(1)}(\varrho_{X^\alpha}) = (X^\alpha)^{\Lambda^\alpha(\beta)} \\ x & \mapsto & \mathcal{F} \overset{\mathbf{L}}{\otimes} \mathbb{C}_x. \end{array} \tag{2.39}$$

Since $\Phi^{\mathcal{F}^\vee[2]}_{X^\alpha\to X} = \Phi^{\mathcal{E}^\alpha}_{X^\alpha\to X}$, we have $\mathcal{C} = \Phi^{\mathcal{F}^\vee[2]}_{X^\alpha\to X}(\operatorname{Per}(X^\alpha/Y, \mathbf{b}^\alpha_1, ..., \mathbf{b}^\alpha_n))$.
(2) *We also have an isomorphism*

$$\begin{array}{rcl} X & \to & M^{(G^\alpha)^\vee, -D_{X^\alpha}\circ\Lambda^\alpha(\beta)}_{\mathcal{O}_{X^\alpha}(1)}(\varrho_{X^\alpha}) \\ x & \mapsto & \mathcal{E}^\alpha \overset{\mathbf{L}}{\otimes} \mathbb{C}_x, \end{array} \tag{2.40}$$

where $M^{(G^\alpha)^\vee, -D_{X^\alpha}\circ\Lambda^\alpha(\beta)}_{\mathcal{O}_{X^\alpha}(1)}(\varrho_{X^\alpha})$ is the moduli of stable objects of $\Lambda^\alpha(\mathcal{C})^D$.
Thus X and X^α are Fourier-Mukai dual.

Proof. (1) is a consequence of Corollary 2.3.6. (2) is a consequence of (1) and the isomorphism $\mathcal{M}^{G^\alpha,\gamma}_{\mathcal{O}_{X^\alpha}(1)}(\varrho_{X^\alpha})^{ss} \to \mathcal{M}^{(G^\alpha)^\vee, -D_{X^\alpha}(\gamma)}_{\mathcal{O}_{X^\alpha}(1)}(\varrho_{X^\alpha})^{ss}$ defined by $E \mapsto D_{X^\alpha}(E)[2]$. ☐

The following proposition explains the condition of the stability of $\mathbb{C}_x$.

Proposition 2.3.18. *$\mathcal{C} = \Lambda^\gamma(\operatorname{Per}(X'/Y, \mathbf{b}_1, ..., \mathbf{b}_n))$ with $X = (X')^\gamma$ if and only if there is a $\beta \in \varrho_X^\perp$ such that $\mathbb{C}_x$ are β-stable for all $x \in X$.*

Proof. For $X = (X')^\gamma$, γ-stability of $\mathcal{E}^\gamma_{|X' \times \{x\}}$ and Corollary 2.3.6 imply the β-stability of $\mathbb{C}_x$, where $\beta := \Lambda^\gamma(\gamma)$. Conversely if $\mathbb{C}_x$ are β-stable for all $x \in X$, then Proposition 2.3.17 (1) implies the claim, where $X' := X^\alpha$ and $\gamma := \Lambda^\alpha(\beta)$. □

We give two examples of $\mathcal{C}$ satisfying the stability condition of $\mathbb{C}_x$.

Lemma 2.3.19. (1) *Assume that* $\mathcal{C} = \mathrm{Per}(X/Y, \mathbf{b}_1, ..., \mathbf{b}_n)$. *If* $-\langle \alpha, v(\mathcal{O}_{C_{ij}}(b_{ij})[1])\rangle > 0$ *for all* $j > 0$, *then* $X \cong X^\alpha$ *by sending* $x \in X$ *to* $\mathbb{C}_x \in X^\alpha$. *Moreover* $A_{p_i} \otimes \mathcal{O}_C$ *such that* $\mathcal{O}_C$ *is a purely 1-dimensional* $\mathcal{O}_{Z_i}$*-module with* $\chi(\mathcal{O}_C) = 1$ *are* α*-stable.*
(2) *Assume that* $\mathcal{C} = \mathrm{Per}(X/Y, \mathbf{b}_1, ..., \mathbf{b}_n)^*$. *If* $-\langle \alpha, v(\mathcal{O}_{C_{ij}}(b_{ij}))\rangle < 0$ *for all* $j > 0$, *then* $X \cong X^\alpha$ *by sending* $x \in X$ *to* $\mathbb{C}_x \in X^\alpha$.

Proof. We only prove (1). Since $\mathbb{C}_x$, $x \in X \setminus \cup_{i=1}^n Z_i$ is irreducible, it is α-twisted stable for any α. For $x \in Z_i$, assume that there is an exact sequence

$$0 \to E_1 \to \mathbb{C}_x \to E_2 \to 0 \tag{2.41}$$

such that $E_1 \neq 0$, $E_2 \neq 0$ and $-\langle \alpha, v(E_1)\rangle = \chi(v^{-1}(\alpha), E_1) \geq 0$. We note that $-\langle \alpha, v(E_{ij})\rangle > 0$ for all $j > 0$. Since $\langle \alpha, \varrho_X\rangle = 0$, $\langle \alpha, v(A_0(\mathbf{b}_i))\rangle = -\sum_{j>0} a_{ij}\langle \alpha, v(E_{ij})\rangle$. As a 0-semi-stable object, E_1 is S-equivalent to $\oplus_{j>0}\mathcal{O}_{C_{ij}}(b_{ij})[1]^{\oplus a'_{ij}}$, $a'_{ij} \leq a_{ij}$. Since $\mathrm{Hom}(\mathcal{O}_{C_{ij}}(b_{ij})[1], \mathbb{C}_x) = 0$, this is impossible. Therefore $\mathbb{C}_x$ is α-twisted stable. Then we have an injective morphism $\phi : X \to X^\alpha$ by sending $x \in X$ to $\mathbb{C}_x$. By using the Fourier-Mukai transform $\Phi^{\mathcal{O}_\Delta^\vee}_{X \to X} : \mathbf{D}(X) \to \mathbf{D}(X)$, we see that ϕ is surjective. Since both spaces are smooth, ϕ is an isomorphism. The last claim also follows by a similar argument. □

2.3.3. *Relation with the twist functor* [S-T]. Let F be a spherical object of $\mathbf{D}(X)$ and set

$$\mathcal{E} := \mathrm{Cone}(F^\vee \boxtimes F \to \mathcal{O}_\Delta)[1]. \tag{2.42}$$

Then $T_F := \Phi^{\mathcal{E}}_{X \to X}$ is an autoequivalence of $\mathbf{D}(X)$.

Lemma 2.3.20. *Let* $\Pi : \mathbf{D}(X) \to \mathbf{D}(Y)$ *be a Fourier-Mukai transform. Then*

$$\Pi \circ T_F \cong T_{\Pi(F)} \circ \Pi. \tag{2.43}$$

Proof. Let $\mathbf{E} \in \mathbf{D}(X \times Y)$ be an object such that $\Pi = \Phi^{\mathbf{E}}_{X \to Y}$. It is sufficient to prove $\Pi(\mathcal{E}) \cong T_{\Pi(F)}(\mathbf{E})$. We set $X_i := X$, $i = 1, 2$. We note that $F^\vee \cong \mathrm{Hom}_p(\mathcal{O}_{X_1} \boxtimes F, \mathcal{O}_\Delta)$, where $p : X_1 \times X_2 \to X_1$ is the projection and $\Delta \subset X_1 \times X_2$ the diagonal. Then

$$\mathcal{E} \cong \mathrm{Cone}(\mathrm{Hom}_p(\mathcal{O}_{X_1} \boxtimes F, \mathcal{O}_\Delta) \boxtimes F \to \mathcal{O}_\Delta)[1]. \tag{2.44}$$

Let $p_{X_2} : Y \times X_2 \to X_2$, $p_Y : Y \times X_2 \to Y$ and $q : X_1 \times Y \to X_1$ be the projections. We have a morphism

$$\begin{aligned}\mathrm{Hom}_p(\mathcal{O}_{X_1} \boxtimes F, \mathcal{O}_\Delta) &\to \mathrm{Hom}_{q'}(\mathcal{O}_{X_1} \boxtimes (\mathbf{E} \otimes p_{X_2}^*(F)), (\mathcal{O}_{X_1} \boxtimes \mathbf{E})_{|\Delta'}) \\ &\to \mathrm{Hom}_q(\mathcal{O}_{X_1} \boxtimes \mathbf{R}p_{Y*}(\mathbf{E} \otimes p_{X_2}^*(F)), \mathbf{E}),\end{aligned} \tag{2.45}$$

where $\Delta' = \Delta \times Y$ and $q' : X_1 \times Y \times X_2 \to X_1$ is the projection. We also have a commutative diagram in $\mathbf{D}(Y \times X_1)$:

$$\begin{array}{ccc} \mathrm{Hom}_p(\mathcal{O}_{X_1} \boxtimes F, \mathcal{O}_\Delta) \boxtimes \Pi(F) & \xrightarrow{\alpha} & \mathbf{E} \\ {\scriptstyle\gamma}\downarrow & & \| \\ \mathrm{Hom}_q(\mathcal{O}_{X_1} \boxtimes \Phi^{\mathbf{E}}_{X \to Y}(F), \mathbf{E}) \boxtimes \Pi(F) & \xrightarrow{\beta} & \mathbf{E}. \end{array} \tag{2.46}$$

Since Π is an equivalence, γ is an isomorphism. Since $\Pi(\mathcal{E}) \cong \mathrm{Cone}(\alpha)[1]$ and $T_{\Pi(F)}(\mathbf{E}) \cong \mathrm{Cone}(\beta)[1]$, we get $\Pi(\mathcal{E}) \cong T_{\Pi(F)}(\mathbf{E})$. □

Corollary 2.3.21. *Assume that* $\mathrm{Supp}(H^i(F)) \subset Z$ *for all* i. *Let* D *be the pull-back of a divisor on* Y. *Then* $T_F(E(D)) \cong T_F(E)(D)$.

Proof. We apply Lemma 2.3.20 to $\Pi = \Phi^{\mathcal{O}_\Delta(D)}_{X \to X}$. Since $\Pi(F) \cong F$, we get our claim. □

Proposition 2.3.22. *Assume that* $G^\vee \otimes G$ *satisfies* $R^1\pi_*(G^\vee \otimes G) = 0$. *Assume that* $G' := T_F(G)$ *is a locally free sheaf up to shift.*

(1) $\mathbf{R}^1\pi_*(G'^\vee \otimes G') = 0$ *and* $\pi_*(G'^\vee \otimes G') \cong \pi_*(G^\vee \otimes G)$.

(2) *We set* $\mathcal{A}' := \pi_*(G'^{\vee} \otimes G')$. *We identify* $\mathrm{Coh}_{\mathcal{A}}(Y)$ *with* $\mathrm{Coh}_{\mathcal{A}'}(Y)$ *via* $\mathcal{A} \cong \mathcal{A}'$. *Then we have a commutative diagram*

$$\begin{array}{ccc} \mathrm{Per}(X/Y, \mathbf{b}_1, ..., \mathbf{b}_n) & \xrightarrow{T_F} & T_F(\mathrm{Per}(X/Y, \mathbf{b}_1, ..., \mathbf{b}_n)) \\ {\scriptstyle \mathbf{R}\pi_*\mathcal{H}om(G,\ )}\downarrow & & \downarrow{\scriptstyle \mathbf{R}\pi_*\mathcal{H}om(G',\ )} \\ \mathrm{Coh}_{\mathcal{A}}(Y) & = & \mathrm{Coh}_{\mathcal{A}'}(Y) \end{array} \tag{2.47}$$

Proof. The proof is almost the same as that of Proposition 2.3.4. □

Definition 2.3.23. For an $\alpha \in H^{\perp} \otimes \mathbb{Q}$, $\mathcal{X}^{\alpha}$ denotes the moduli stack of α-semi-stable objects E of $\mathcal{C}$ such that $v(E) = \varrho_X$.

For an $\alpha \in H^{\perp} \otimes \mathbb{Q}$, let F be an α-stable object such that (i) $\langle v(F)^2 \rangle = -2$ and (ii) $\langle \alpha, v(F) \rangle = 0$. By (i), F is a spherical object. By the same proof of [O-Y, Prop. 1.12], we have the following result.

Proposition 2.3.24. *We set* $\alpha^{\pm} := \pm\epsilon v(F) + \alpha$, *where* $0 < \epsilon \ll 1$. *Then* T_F *induces an isomorphism*

$$\begin{array}{ccc} \mathcal{X}^{\alpha^-} & \to & \mathcal{X}^{\alpha^+} \\ E & \mapsto & T_F(E) \end{array} \tag{2.48}$$

which preserves the S*-equivalence classes. Hence we have an isomorphism*

$$X^{\alpha^-} \to X^{\alpha^+}. \tag{2.49}$$

Combining Proposition 2.3.24 with Lemma 2.3.20, we get the following corollary.

Corollary 2.3.25. *Assume that* α *belongs to exactly one wall defined by* F. *Then* T_F *induces an isomorphism* $X^{\alpha^-} \to X^{\alpha^+}$. *Under this isomorphism, we have*

$$\Phi_{X^{\alpha^-} \to X}^{\mathcal{E}^{\alpha^+}} \cong T_F \circ \Phi_{X^{\alpha^-} \to X}^{\mathcal{E}^{\alpha^-}} \cong \Phi_{X^{\alpha^-} \to X}^{\mathcal{E}^{\alpha^-}} \circ T_A, \tag{2.50}$$

where $A := \Phi_{X \to X^{\alpha^-}}^{(\mathcal{E}^{\alpha^-})^{\vee}[2]}(F)$.

2.4. **Construction of a local projective generator.**

We return to the general situation in section 2.1. We shall construct local projective generators for $\mathrm{Per}(X/Y, \{L_{ij}\})$.

Proposition 2.4.1. *Let* β *be a 2-cocycle of* $\mathcal{O}_X^{\times}$ *defining a torsion element of* $H^2(X, \mathcal{O}_X^{\times})$. *Assume that* $E \in K^{\beta}(X)$ *satisfies*

$$\begin{aligned} 0 \le -\chi(E, L_{ij}),\ 1 \le j \le s_i, \\ -\sum_j a_{ij}\chi(E, L_{ij}) \le r \end{aligned} \tag{2.51}$$

for all i.

(1) *There is a locally free* β*-twisted sheaf* G *on* X *such that* $R^1\pi_*(G^{\vee} \otimes G) = 0$, $\mathbf{R}\pi_*(G^{\vee} \otimes F) \in \mathrm{Coh}(Y)$ *for* $F \in \mathrm{Per}(X/Y, \{L_{ij}\})$, G *is* μ*-stable and* $\tau(G) = \tau(E) - n\tau(\mathbb{C}_x)$, $n \gg 0$.
(2) *There is a locally free* β*-twisted sheaf* G *on* X *such that* $R^1\pi_*(G^{\vee} \otimes G) = 0$, $\mathbf{R}\pi_*(G^{\vee} \otimes F) \in \mathrm{Coh}(Y)$ *for* $F \in \mathrm{Per}(X/Y, \{L_{ij}\})$ *and* $\tau(G) = 2\tau(E)$.
(3) *Moreover if the inequalities in* (2.51) *are strict, then* G *in (1) and (2) are local projective generators of* $\mathrm{Per}(X/Y, \{L_{ij}\})$.

Corollary 2.4.2. *Assume that* $(r, \xi) \in \mathbb{Z}_{>0} \oplus \mathrm{NS}(X)$ *satisfies*

$$\begin{aligned} 0 < (\xi, C_{ij}) - r(b_{ij} + 1),\ 1 \le j \le s_i, \\ \sum_j a_{ij}(\xi, C_{ij}) - r\sum_j a_{ij}(b_{ij} + 1) < r, \end{aligned} \tag{2.52}$$

for all i.

(1) *For any sufficiently large* n, *there is a local projective generator* G *of* $\mathrm{Per}(X/Y, \mathbf{b}_1, ..., \mathbf{b}_n)$ *such that* G *is a* μ*-stable sheaf with respect to* H *and* $(\mathrm{rk}\, G, c_1(G), c_2(G)) = (r, \xi, c_2)$.
(2) *For any* $\mathbf{e} \in K(X)_{\mathrm{top}}$ *with* $(\mathrm{rk}\,\mathbf{e}, c_1(\mathbf{e})) = (r, \xi)$, *there is a local projective generator* G *such that* $\tau(G) = 2\mathbf{e}$.

Proof of Proposition 2.4.1.

(1) We assume that H is represented by a smooth connected curve with $Z \cap H = \emptyset$, where $Z = \sum_{i=1}^n Z_i$. We take a torsion free sheaf E such that $\mathrm{Ext}^2(E, E(-Z - H))_0 = 0$. By the construction of E, we may assume that E is locally free on $Z \cup H$. We consider the restriction morphism of the local deformation spaces

$$\phi : \mathrm{Def}(X, E) \to \mathrm{Def}(Z, E_{|Z}) \times \mathrm{Def}(H, E_{|H}). \tag{2.53}$$

Then $\mathrm{Def}(X,E)$ and $\mathrm{Def}(Z,E_{|Z})\times \mathrm{Def}(H,E_{|H})$ are smooth, and ϕ is submersive. In particular, by using Lemma 2.4.3 below, we see that E deforms to a locally free β-twisted sheaf G such that G is μ-stable with respect to H and $\mathrm{Hom}(G,L_{ij})=\mathrm{Ext}^1(G,A_{p_i})=0$ for all i,j. By Remark 1.1.23, Proposition 2.4.1 (1) holds.

(2) By (1), we have locally free sheaves E_i, $i=1,2$ such that $R^1\pi_*(E_i^\vee\otimes E_i)=0$, $\mathbf{R}\pi_*(G_i^\vee\otimes F)\in \mathrm{Coh}(Y)$ for $F\in \mathrm{Per}(X/Y,\{L_{ij}\})$, $\tau(E_i)=\tau(E)-n_i\tau(\mathbb{C}_x)$ and $n_1+n_2=n^2(H^2)\operatorname{rk} E$. Then $G=E_1(nH)\oplus E_2(-nH)$ satisfies the claim.

(3) The claim follows from Proposition 1.1.22. □

Lemma 2.4.3. (1) *$E_{|Z}$ deforms to a locally free β-twisted sheaf such that*

$$H^0(C_{ij},E^\vee\otimes L_{ij})=H^1(Z_i,E^\vee\otimes A_{p_i})=0 \tag{2.54}$$

for all i,j.

(2) *$E_{|H}$ deforms to a μ-stable locally free β-twisted sheaf on H.*

Proof. (1) Since $E_{|Z}=\oplus_{i=1}^n E_{|Z_i}$, we shall prove the claims for each $E_{|Z_i}$. Since $H^2(Z,\mathcal{O}_Z^\times)=\{1\}$, there is a β-twisted line bundle $\mathcal{L}$ on Z_i which induces an equivalence $\varphi:\mathrm{Coh}^\beta(Z)\cong \mathrm{Coh}(Z)$ in (1.134). Since $\mathrm{Pic}(Z_i)\to \mathbb{Z}^{s_i}$ ($L\mapsto \prod_{j=1}^{s_i}\deg(L_{|C_{ij}})$) is an isomorphism, we may assume that $\varphi(L_{ij})=\mathcal{O}_{C_{ij}}(-1)$. Thus we may assume that β is trivial and $L_{ij}=\mathcal{O}_{C_{ij}}(-1)$. In this case, we have $A_{p_i}=\mathcal{O}_{Z_i}$. Then we have $\deg(E_{|C_{ij}})\geq 0$ for all $j>0$ and $\deg(E_{|Z_i})\leq r$. Let D be an effective Cartier divisor on Z_i such that $(D,C_{ij})=\deg(E_{|C_{ij}})$. Then

$$K:=\ker(H^0(\mathcal{O}_{Z_i\cap D})\otimes\mathcal{O}_{Z_i}\to\mathcal{O}_{Z_i\cap D}) \tag{2.55}$$

is a locally free sheaf on Z_i such that $H^1(Z_i,K)=0$ and $H^0(C_{ij},K_{|C_{ij}}(-1))=0$. Since $\operatorname{rk} K=\dim H^0(\mathcal{O}_{Z_i\cap D})=\deg_{Z_i}(D)=\deg(E_{|Z_i})\leq r$, we set $F:=K^\vee\oplus\mathcal{O}_{Z_i}^{\oplus(\operatorname{rk} E-\operatorname{rk} K)}$. Since F is a locally free sheaf with $(\operatorname{rk} F,\det(F^\vee))=(\operatorname{rk} E_{|Z_i},\det(E_{|Z_i}))$, we get the claim by Lemma 2.1.4 and the openness of the condition (2.54).

(2) is well-known. □

Corollary 2.4.4. *Assume that π is the minimal resolution of rational double points $p_1,...,p_n$. Let $\mathcal{C}$ be the category in Lemma 1.1.5 and E_{ij}, $1\leq i\leq n$, $0\leq j\leq s_i$ the 0-stable objects in Lemma 2.2.12 (2). For an element $E\in K(X)$ satisfying $\chi(E,E_{ij})>0$ for all i,j, there is a local projective generator G of $\mathcal{C}$ such that $\tau(G)=2\tau(E)$.*

Proof. We consider the equivalence Λ^α in Proposition 2.3.16. Then since $\chi(\Lambda^\alpha(E),\Lambda^\alpha(E_{ij}))>0$ for all i,j, Proposition 2.4.1 implies that there is a local projective generator G^α of $\Lambda^\alpha(\mathcal{C})$ such that $\tau(G^\alpha)=2\tau(\Lambda^\alpha(E))$. We set $G:=(\Lambda^\alpha)^{-1}(G^\alpha)\in\mathcal{C}$. Then

$$\begin{aligned} H^0(X,H^k(G\overset{\mathbf{L}}{\otimes}\mathbb{C}_x)) =& H^k(X,G\overset{\mathbf{L}}{\otimes}\mathbb{C}_x)\\ =& \mathrm{Hom}(\mathbb{C}_x,G[k+2])\\ =& \mathrm{Hom}(\Lambda^\alpha(\mathbb{C}_x),G^\alpha[k+2])\\ =& \mathrm{Hom}(G^\alpha,\Lambda^\alpha(\mathbb{C}_x)[-k])=0 \end{aligned} \tag{2.56}$$

for all $x\in X$ and $k\neq 0$. Therefore G is a locally free sheaf on X. Since G^α is a local projective generator of $\Lambda^\alpha(\mathcal{C})$ and Λ^α is an equivalence, G is a local projective generator of $\mathcal{C}$. □

2.4.1. *More results on the structure of $\mathcal{C}$.* Let $\mathcal{C}$ be the category of perverse coherent sheaves in Lemma 1.1.5. Assume that there is $\beta\in \mathrm{NS}(X)\otimes\mathbb{Q}$ such that $\mathbb{C}_x$ is β-stable for all $x\in X$. By Proposition 2.3.18, $\mathcal{C}=\Lambda^\alpha(\mathrm{Per}(X'/Y,\mathbf{b}_1,...,\mathbf{b}_n))$. So we first assume that $\mathcal{C}=\mathrm{Per}(X/Y,\mathbf{b}_1,...,\mathbf{b}_n)$ and set

$$E_{ij}:=\begin{cases}\mathcal{O}_{C_{ij}}(b_{ij})[1], & j>0,\\ A_0(\mathbf{b}), & j=0.\end{cases} \tag{2.57}$$

We set $v_{ij}:=v(E_{ij})$. Let u_0 be an isotropic Mukai vector such that $r_0:=\operatorname{rk} u_0>0$, $\langle u_0,v_{ij}\rangle=0$ for all i,j. We set

$$L:=\mathbb{Z}u_0+\sum_{i=1}^n\sum_{j=0}^{s_i}\mathbb{Z}v_{ij}. \tag{2.58}$$

Then L is a sublattice of $H^*(X,\mathbb{Z})$ and we have a decomposition

$$L=(\mathbb{Z}u_0\oplus\mathbb{Z}\varrho_X)\perp(\oplus_{i=1}^n\oplus_{j=1}^{s_i}\mathbb{Z}v_{ij}). \tag{2.59}$$

We set

$$\begin{aligned} T_i:=&\oplus_{j=1}^{s_i}\mathbb{Z}C_{ij},\\ T:=&\oplus_{i=1}^n T_i. \end{aligned} \tag{2.60}$$

Then we have an isometry

$$\psi: \quad \begin{array}{ccc} \oplus_{i=1}^n \oplus_{j=1}^{s_i} \mathbb{Z}v_{ij} & \to & T \\ v & \mapsto & c_1(v). \end{array} \tag{2.61}$$

Combining the isometry $\mathbb{Z}u_0 \oplus \mathbb{Z}\varrho_X \to \mathbb{Z}r_0 \oplus \mathbb{Z}\varrho_X$ $(xu_0 + z\varrho_X \mapsto xr_0 + z\varrho_X)$, we also have an isometry

$$\widetilde{\psi} : (\mathbb{Z}u_0 \oplus \mathbb{Z}\varrho_X) \perp (\oplus_{i=1}^n \oplus_{j=1}^{s_i} \mathbb{Z}v_{ij}) \to (\mathbb{Z}r_0 \oplus \mathbb{Z}\varrho_X) \perp T \tag{2.62}$$

Let $\mathfrak{g}_i$ (resp. $\widehat{\mathfrak{g}}_i$) be the finite Lie algebra (resp. affine Lie algebra) associated to the lattice $\oplus_{j=1}^{s_i}\mathbb{Z}v_{ij}$ (resp. $\oplus_{j=0}^{s_i}\mathbb{Z}v_{ij}$). Let $\mathfrak{g}$ (resp. $\widehat{\mathfrak{g}}$) be the Lie algebra associated to $\oplus_{i=1}^n \oplus_{j=1}^{s_i} \mathbb{Z}v_{ij}$ (resp. $\oplus_{i=1}^n \oplus_{j=0}^{s_i} \mathbb{Z}v_{ij}$).

Let $W(\mathfrak{g}_i)$ (resp. $W(\mathfrak{g})$) be the Weyl group of $\mathfrak{g}_i$ (resp. $\mathfrak{g}$) and $\mathcal{W}_i$ (resp. $\mathcal{W}$) the set of Weyl chambers of $W(\mathfrak{g}_i)$ (resp. $W(\mathfrak{g})$). Since $\mathfrak{g} = \oplus_{i=1}^n \mathfrak{g}_i$, $W(\mathfrak{g}) = \prod_{i=1}^n W(\mathfrak{g}_i)$ and $\mathcal{W} = \prod_{i=1}^n \mathcal{W}_i$. By the action of $W(\mathfrak{g})$, $\mathbb{Q}u_0 + \mathbb{Q}\varrho_X$ is fixed. Let $W(\widehat{\mathfrak{g}}_i)$ (resp. $W(\widehat{\mathfrak{g}})$) be the Weyl group of $\widehat{\mathfrak{g}}_i$ (resp. $\widehat{\mathfrak{g}}$). We have the following decompositions

$$\begin{aligned} W(\widehat{\mathfrak{g}}_i) =& T_i \rtimes W(\mathfrak{g}_i), \\ W(\widehat{\mathfrak{g}}) =& T \rtimes W(\mathfrak{g}), \end{aligned} \tag{2.63}$$

and the action of $D \in T$ on L is the multiplication by e^D. Indeed

$$T_{\mathcal{O}_{C_{ij}}(b_{ij}+1)} \circ T_{\mathcal{O}_{C_{ij}}(b_{ij})[1]} = e^{-C_{ij}}$$

as an isometry of L.

We shall study the category $\Lambda^\alpha(\mathcal{C})$. We may assume that $\alpha \in \mathrm{NS}(X) \otimes \mathbb{Q}$ is $\alpha = \sum_i \alpha_i$ with $\alpha_i \in T_i \otimes \mathbb{Q}$. Via the identification ψ, we have an action of W on $T \otimes \mathbb{Q}$. We set

$$\begin{aligned} C_i^{\mathrm{fund}} :=& \{\alpha \in T_i \otimes \mathbb{R} | (\alpha, C_{ij}) > 0, 1 \le j \le s_i\}, \\ C^{\mathrm{fund}} :=& \prod_{i=1}^n C_i^{\mathrm{fund}}. \end{aligned} \tag{2.64}$$

C^{fund} is the fundamental Weyl chamber. If $\alpha \in C^{\mathrm{fund}}$, then Lemma 2.3.19 implies that $\mathbb{C}_x$ is α-stable for all $x \in X$. By the action of $W(\mathfrak{g}_i)$, we have $\mathcal{W}_i = W(\mathfrak{g}_i)C_i^{\mathrm{fund}}$. We also set

$$C_{\mathrm{alcove}}^{\mathrm{fund}} := \{\alpha \in T \otimes \mathbb{R} | (\alpha, C_{ij}) > 0, 1 \le j \le s_i,\ (\alpha, Z_i) < 1\}. \tag{2.65}$$

By the isometry $\widetilde{\psi}^{-1}$, we have

$$\begin{aligned} (\alpha, C_{ij}) =& -\langle \psi^{-1}(\alpha), v_{ij} \rangle \\ =& -\langle (\frac{u_0}{\operatorname{rk} u_0} + \psi^{-1}(\alpha) + \frac{(\alpha^2)}{2}\varrho_X), v_{ij} \rangle = -\langle e^{\frac{c_1(u_0)}{\operatorname{rk} u_0}+\alpha}, v_{ij} \rangle \end{aligned} \tag{2.66}$$

for $j > 0$ and $1 - (\alpha, Z_i) = 1 + \sum_{j=1}^{s_i} a_{ij} \langle e^{\frac{c_1(u_0)}{\operatorname{rk} u_0}+\alpha}, v_{ij} \rangle = -\langle e^{\frac{c_1(u_0)}{\operatorname{rk} u_0}+\alpha}, v_{i0} \rangle$. Hence we have

$$C_{\mathrm{alcove}}^{\mathrm{fund}} = \{\alpha \in T \otimes \mathbb{R} | -\langle e^{\frac{c_1(u_0)}{\operatorname{rk} u_0}+\alpha}, v_{ij} \rangle > 0\}. \tag{2.67}$$

Applying Corollary 2.3.25 successively, we get the following result.

Proposition 2.4.5. *If $\alpha \in T \otimes \mathbb{Q}$ belongs to a chamber $C = \prod_{i=1}^n C_i$, $C_i \subset T_i \otimes \mathbb{Q}$, then there are rigid objects $F_1, ..., F_n \in \mathcal{C}$ such that $X^\alpha \cong X$ and $\Phi_{X \to X}^{\mathcal{E}^\alpha} = T_{F_n} \circ T_{F_{n-1}} \circ \cdots \circ T_{F_1}$. Thus $\Lambda^\alpha = (\Phi_{X \to X}^{\mathcal{E}^\alpha})^{-1}$ induces an isometry $w(\alpha)$ of L.*

Then we have a map

$$\phi: \quad \begin{array}{ccc} \mathcal{W} & \to & W(\widehat{\mathfrak{g}})/T \\ C(\alpha) & \mapsto & [w(\alpha) \mod T], \end{array} \tag{2.68}$$

where $C(\alpha)$ is the chamber containing α.

Lemma 2.4.6. *$\phi : \mathcal{W} \to W(\widehat{\mathfrak{g}})/T \cong W(\mathfrak{g})$ is bijective.*

Proof. There is an element α_0 in the fundamental Weyl chamber such that $\alpha = \Phi_{X \to X}^{\mathcal{E}^\alpha}(\alpha_0)$. Hence $w(\alpha)(C(\alpha)) = C(\alpha_0)$. Thus ϕ is injective. Since $\#\mathcal{W}_i = \#W(\mathfrak{g}_i)$, ϕ is bijective. ☐

We set

$$T^* := \{D \in T \otimes \mathbb{Q} | (D, C_{ij}) \in \mathbb{Z}\}. \tag{2.69}$$

Then $\widetilde{W} := T^* \rtimes W(\mathfrak{g})$ is the extended Weyl group. By the action of $\widetilde{W}$, we can change $(\mathbf{b}_1, ..., \mathbf{b}_n)$ to any sequence $(\mathbf{b}'_1, ..., \mathbf{b}'_n)$.

Proposition 2.4.7. *Let $\mathcal{C}$ be the category in Lemma 1.1.5 and assume that there is $\beta \in \mathrm{NS}(X)\otimes\mathbb{Q}$ such that $\mathbb{C}_x$ is β-stable for all $x \in X$. Then $\mathcal{C}$ is equivalent to ${}^{-1}\operatorname{Per}(X/Y)$. In particular, $\operatorname{Per}(X/Y, \mathbf{b}_1, ..., \mathbf{b}_n) \cong {}^{-1}\operatorname{Per}(X/Y)$.*

Proof. We may assume that $\mathcal{C} = \operatorname{Per}(X/Y, \mathbf{b}_1, ..., \mathbf{b}_n)$. We set

$$u_{ij} := \begin{cases} v(\mathcal{O}_{C_{ij}}(-1)[1]), & j > 0, \\ v(\mathcal{O}_{Z_i}), & j = 0. \end{cases} \tag{2.70}$$

By the theory of affine Lie algebras, there is an element $w \in W(\widehat{\mathfrak{g}})$ such that

$$\begin{aligned} &w(\{\beta \in T\otimes\mathbb{R} \mid -\langle e^{\beta}, v_{ij}\rangle > 0, i, j \geq 0\}) \\ =&\{\beta \in T\otimes\mathbb{R} \mid -\langle e^{\beta}, u_{ij}\rangle > 0, i, j \geq 0\}. \end{aligned} \tag{2.71}$$

Then we have

$$\{w(v_{ij}) | 0 \leq j \leq s_i\} = \{u_{ij} | 0 \leq j \leq s_i\}$$

for all i.

For each i, there is an integer j_i such that (1) $c_1(w(v_{ij_i}))$ is effective and (2) $-c_1(w(v_{ij}))$, $j \neq j_i$ are effective. By Lemma 2.4.6, we have $w = e^D\phi(\alpha)$, $D, \alpha \in T$. Since $v(\Lambda^{\alpha}(E_{ij})\otimes\mathcal{O}_X(D)) = e^D v(\Lambda^{\alpha}(E_{ij})) = e^D\phi(\alpha)(v_{ij})$, Proposition 2.3.4 (2) implies that $-(\alpha, c_1(E_{ij})) > 0$ unless $j = j_i$. By Lemma 2.2.18 and Lemma 2.3.11, $\Lambda^{\alpha}(E_{ij})[-1]$, $j \neq j_i$ is a line bundle on a smooth rational curve and $\Lambda^{\alpha}(E_{ij_i})$ is a line bundle on Z_i. Thus

$$\begin{aligned} \{\Lambda^{\alpha}(E_{ij})\otimes\mathcal{O}_X(D) | j \neq j_i\} =&\{\mathcal{O}_{C_{ij}}(-1)[1] | 0 < j \leq s_i\}, \\ \Lambda^{\alpha}(E_{ij_i})\otimes\mathcal{O}_X(D) =&\mathcal{O}_{Z_i}. \end{aligned} \tag{2.72}$$

By Proposition 2.3.4 (5), we get $\Lambda^{\alpha}(\mathcal{C})\otimes\mathcal{O}_X(D) \cong^{-1} \operatorname{Per}(X/Y)$. □

Remark 2.4.8. For the derived category of coherent twisted sheaves, we also see that the equivalence classes of $\operatorname{Per}(X/Y, \{L_{ij}\})$ does not depend on the choice of $\{L_{ij}\}$.

Proposition 2.4.9. *We set $v = (r, \xi, a) \in H^{ev}(X, \mathbb{Z})_{\mathrm{alg}}$, $r > 0$. Assume that $(\xi, D) \notin r\mathbb{Z}$ for all $D \in T$ with $(D^2) = -2$. Then there is a category of perverse coherent sheaves $\mathcal{C}_v$ and a locally free sheaf G on X such that G is a local projective generator of $\mathcal{C}_v$ with $v(G) = 2v$. We also have a local projective generator G' of $\mathcal{C}_v$ such that G' is μ-stable with respect to H and $v(G') = v - b\varrho_X$, $b \gg 0$.*

Proof. We set $\mathcal{C} = \operatorname{Per}(X/Y, \mathbf{b}_1, ..., \mathbf{b}_n)$ and keep the notation as above. By our assumption, $\langle v, u\rangle \notin r\mathbb{Z}$ for all (-2)-vectors $u \in L$. Then there is $w \in W$ such that $v = w(v_f)$ and v_f/r belongs to the fundamental alcove, that is, $-\langle v_f/r, v_{ij}\rangle > 0$ for all i, j. By Lemma 2.4.6, we have an element α such that $w = e^D\phi(\alpha)$, $D \in T$. By Proposition 2.4.1, there is a local projective generator G_f of $\mathcal{C}$ such that $v(G_f) = 2v_f$. We set $\mathcal{C}_v := \Lambda^{\alpha}(\mathcal{C})\otimes\mathcal{O}_X(D)$. Then $G^{\alpha} := \Lambda^{\alpha}(G_f)$ is a local projective generator of $\mathcal{C}_v\otimes\mathcal{O}_X(-D)$. Hence $G := G^{\alpha}(D)$ is a local projective generator of $\mathcal{C}_v$ such that $v(G) = 2v$. □

2.5. **Deformation of a local projective generator.** Let $f : (\mathcal{X}, \mathcal{L}) \to S$ be a flat family of polarized surfaces over S. For a point $s_0 \in S$, we set $X := \mathcal{X}_{s_0}$. Let $\mathcal{H}$ be a relative Cartier divisor on X such that $H := \mathcal{H}_{s_0}$ gives a contraction $f : X \to Y$ to a normal surface Y with $\mathbf{R}\pi_*(\mathcal{O}_X) = \mathcal{O}_Y$. We shall construct a family of contractions $f : \mathcal{X} \to \mathcal{Y}$ over a neighborhood of s_0.

Replacing H by mH, we may assume that $H^i(X, \mathcal{O}_X(mH)) = H^i(Y, \mathcal{O}_Y(mH)) = 0$ for $m > 0$. We shall find an open neighborhood S_0 of s_0 such that $R^if_*(\mathcal{O}_{\mathcal{X}_{S_0}}(m\mathcal{H})) = 0$, $i > 0, m > 0$ and $f_*(\mathcal{O}_{\mathcal{X}_{S_0}}(m\mathcal{H}))$ is locally free: We consider the exact sequence

$$0 \to \mathcal{O}_{\mathcal{X}}(m\mathcal{H}) \to \mathcal{O}_{\mathcal{X}}((m+1)\mathcal{H}) \to \mathcal{O}_{\mathcal{H}}((m+1)\mathcal{H}) \to 0. \tag{2.73}$$

Since $\mathcal{H} \to S$ is a flat morphism, the base change theorem implies that $R^if_*(\mathcal{O}_{\mathcal{X}}(m\mathcal{H})) \to R^if_*(\mathcal{O}_{\mathcal{X}}((m+1)\mathcal{H}))$ is surjective, if $(m+1)(H^2) > (H^2) + (H, K_X)$. We take an open neighborhood S_0 of s_0 such that $R^if_*(\mathcal{O}_{\mathcal{X}_{S_0}}(m\mathcal{H})) = 0$, $i > 0, (H, K_X)/(H^2) \geq m > 0$. Then the claim holds. We replace S by S_0 and set $\mathcal{Y} := \operatorname{Proj}(\oplus_m f_*(\mathcal{O}_{\mathcal{X}}(m\mathcal{H})))$. Then $\mathcal{Y}$ is flat over S and $\mathcal{Y}_{s_0} \cong Y$. By the construction, $\mathcal{Y} \to S$ is a flat family of normal surfaces.

Let $\mathcal{Z} := \{x \in \mathcal{X} | \dim \pi^{-1}(\pi(x)) \geq 1\}$ be the exceptional locus. Then $\{(\mathcal{Z}_s, \mathcal{L}_s) | s \in S\}$ is a bounded set. Hence $\mathcal{D} := \{D \in \mathrm{NS}(\mathcal{X}_s) | s \in S, (D, \mathcal{H}_s) = 0\}$ is a finite set. Replacing S by an open neighborhood of s_0, we may assume that $D \in \mathcal{D}$ is a deformation of $D_0 \in \mathrm{NS}(X)$ (i.e., D belongs to $\mathrm{NS}(X)$ via the identification $H^2(\mathcal{X}_s, \mathbb{Z}) \cong H^2(X, \mathbb{Z})$).

Lemma 2.5.1. *Assume that there is a locally free sheaf G on $\mathcal{X}$ such that $R^1\pi_*(G^{\vee}\otimes G) = 0$ and $\operatorname{rk} G \nmid (c_1(G)_{s_0}, D)$ for all (-2)-curves with $(D, \mathcal{H}_{s_0}) = 0$. Then replacing S by an open neighborhood of s_0, we may assume that $\operatorname{rk} G \nmid (c_1(G)_s, D)$ for all (-2)-curves with $(D, \mathcal{H}_s) = 0$. Thus G is a family of tilting generators.*

As an example, we consider a family of $K3$ surfaces. Let X be a $K3$ surface and $\pi : X \to Y$ a contraction. Let p_i, $i = 1, 2, ..., n$ be the singular points and $Z_i := \sum_j a_{ij} C_{ij}$ their fundamental cycles. Let H be the pull-back of an ample divisor on Y. Assume that $(r, \xi) \in \mathbb{Z}_{>0} \times \mathrm{NS}(X)$ satisfies $r \nmid (\xi, D)$ for all (-2)-curves D with $(D, H) = 0$. By Proposition 2.4.9, there is a category of perverse coherent sheaves $\mathcal{C}$ and a local projective generator G of $\mathcal{C}$ such that G is μ-stable with respect to H and $(\operatorname{rk} G, c_1(G)) = (r, \xi)$. Replacing G by $G \otimes L^{\otimes m}$, $L \in \mathrm{Pic}(X)$ and $\mathcal{C}$ by $\mathcal{C} \otimes L^{\otimes m}$, we assume that ξ is ample. If $(\mathbb{Q}\xi + \mathbb{Q}H) \cap H^{\perp}$ does not contain a (-2)-curve, then we have a deformation $(\mathcal{X}, \mathcal{L}) \to S$ of (X, ξ) such that $\mathcal{H}_s$ is ample for a general $s \in S$. Since G is simple, replacing S by a smooth covering $S' \to S$, we also have a deformation $\mathcal{G}$ of G over S. By shrinking S, we may assume that $\mathcal{G}$ is a family of tilting generators. Then we can construct a family of moduli spaces $f : \overline{M}_{(\mathcal{X},\mathcal{H})/S}^{\mathcal{G}}(v) \to S$ of $\mathcal{G}_s$-twisted semi-stable objects on $\mathcal{X}_s$, $s \in S$ (for the twisted cases, see Step 3, 4 of the proof of [Y4, Thm. 3.16]). By our assumption, a general fiber of f is the moduli space of $\mathcal{G}_s$-twisted semi-stable sheaves, which is non-empty by Lemma 6.2.3. Hence we get the following lemma.

Lemma 2.5.2. *Assume that v is primitive and $\langle v^2 \rangle \geq -2$. Then f is surjective. In particular, $\overline{M}_{(\mathcal{X},\mathcal{H})/S}^{\mathcal{G}}(v)_{s_0} \neq \emptyset$.*

Remark 2.5.3. We note that $R := \{C \in \mathrm{NS}(X) | (C, H) = 0, (C^2) = -2\}$ is a finite set. If $\rho(X) \geq 3$, then $\cup_{C \in R} (\mathbb{Q}H + \mathbb{Q}C)$ is a proper subset of $\mathrm{NS}(X) \otimes \mathbb{Q}$. Hence $(\mathbb{Q}\xi + \mathbb{Q}H) \cap R = \emptyset$ for a general ξ. In general, we have a deformation $(\mathcal{X}, \mathcal{L}) \to S$ of (X, ξ) such that $\mathcal{G}$ is a family of tilting generators and $\rho(\mathcal{X}_s) \geq 3$ for infinitely many points $s \in S$.

Remark 2.5.4. By the usual deformation theory of objects, we note that $M_{(\mathcal{X},\mathcal{H})/S}^{\mathcal{G}}(v) \to S$ is a smooth morphism. If $\overline{M}_{(\mathcal{X},\mathcal{H})/S}^{\mathcal{G}}(v)_{s_0} = M_{(\mathcal{X},\mathcal{H})/S}^{\mathcal{G}}(v)_{s_0}$, then we have a smooth deformation $\overline{M}_{(\mathcal{X},\mathcal{H})/S}^{\mathcal{G}}(v) \to S$ of $\overline{M}_{(\mathcal{X},\mathcal{H})/S}^{\mathcal{G}}(v)_{s_0}$. In particular, $\overline{M}_{(\mathcal{X},\mathcal{H})/S}^{\mathcal{G}}(v)_{s_0}$ deforms to a usual moduli of semi-stable sheaves.

Corollary 2.5.5. *Let $v_0 = (r, \xi, a)$ be a primitive isotropic Mukai vector such that $r \not| (\xi, D)$ for all (-2)-curves D with $(D, H) = 0$. Let $\mathcal{C}$ be the category in Proposition 2.4.9. Then $M_H^{v_0}(v_0) \neq \emptyset$.*

Proof. By Lemma 2.5.2 and Remark 2.5.3, we see that $\overline{M}_H^{v_0}(v_0) \neq \emptyset$. By the same proof of [O-Y, Lem. 2.17], we see that $\overline{M}_H^{v_0+\alpha}(v_0) \neq \emptyset$ for a general α. Then $\overline{M}_H^{v_0+\alpha}(v_0)$ is a $K3$ surface. In the same way as in the proof of [O-Y, Prop. 2.11], we see that $M_H^{v_0}(v_0) \neq \emptyset$. □

3. Fourier-Mukai transform on a $K3$ surface.

3.1. **Basic results on the moduli spaces of dimension 2.** Let Y be a normal $K3$ surface and $\pi : X \to Y$ the minimal resolution. Let $p_1, p_2, \ldots, p_n$ be the singular points of Y and $Z_i := \pi^{-1}(p_i) = \sum_{j=0}^{s_i} a_{ij} C_{ij}$ the fundamental cycle, where C_{ij} are smooth rational curves on X and $a_{ij} \in \mathbb{Z}_{>0}$. We shall study moduli of stable objects in the category $\mathcal{C}$ in Lemma 1.1.5 satisfying the following assumption.

Assumption 3.1.1. There is a $\beta \in \varrho_X^{\perp} \otimes \mathbb{Q}$ such that $\mathbb{C}_x$ is β-stable for all $x \in X$.

By Proposition 2.3.18, there are $\mathbf{b}_i := (b_{i1}, b_{i2}, \ldots, b_{is_i}) \in \mathbb{Z}^{\oplus s_i}$ and an autoequivalence $\Phi_{X \to X}^{\mathcal{F}^{\vee}[2]} : \mathbf{D}(X) \to \mathbf{D}(X)$ such that $\Phi_{X \to X}^{\mathcal{F}^{\vee}[2]}(\mathrm{Per}(X/Y)) = \mathcal{C}$, where $\mathrm{Per}(X/Y) := \mathrm{Per}(X/Y, \mathbf{b}_1, \ldots, \mathbf{b}_n)$ and $\mathcal{F}$ is the family of $\Phi_{X \to X}^{\mathcal{F}}(\beta)$-stable objects of $\mathrm{Per}(X/Y)$ in Proposition 2.3.18. We set

$$A_{ij} := \begin{cases} \Phi_{X \to X}^{\mathcal{F}^{\vee}[2]}(A_0(\mathbf{b}_i)), & j = 0, \\ \Phi_{X \to X}^{\mathcal{F}^{\vee}[2]}(\mathcal{O}_{C_{ij}}(b_{ij})[1]), & j > 0. \end{cases} \tag{3.1}$$

Throughout this section, we assume the following:

Assumption 3.1.2. $v_0 := r_0 + \xi_0 + a_0 \varrho_X$, $r_0 > 0, \xi_0 \in \mathrm{NS}(X)$ is a primitive isotropic Mukai vector such that $\langle v_0, v(A_{ij}) \rangle < 0$ for all i, j.

By Corollary 2.4.4, we have the following.

Lemma 3.1.3. *There is a local projective generator G of $\mathcal{C}$ whose Mukai vector is $2v_0$. More generally, for a sufficiently small $\alpha \in (v_0^{\perp} \cap \varrho_X^{\perp}) \otimes \mathbb{Q}$, there is a local projective generator G of $\mathcal{C}$ such that $v(G) \in \mathbb{Q}_{>0}(v_0 + \alpha)$.*

Let H be the pull-back of an ample divisor on Y. For a sufficiently small $\alpha \in (v_0^{\perp} \cap \varrho_X^{\perp}) \otimes \mathbb{Q}$, we take a local projective generator G of $\mathcal{C}$ with $v(G) \in \mathbb{Q}_{>0}(v_0 + \alpha)$. We define $v_0 + \alpha$-twisted semi-stability in a usual way. Since it is equivalent to the G-twisted semi-stability, we have the moduli space $\overline{M}_H^{v_0+\alpha}(v_0)$. Let $M_H^{v_0+\alpha}(v_0)$ be the moduli space of $v_0 + \alpha$-stable objects. By Corollary 2.5.5, $M_H^{v_0}(v_0) \neq \emptyset$. Hence we see that $M_H^{v_0+\alpha}(v_0)$ is also non-empty. Then we have the following which is well-known for the moduli of stable sheaves on $K3$ surfaces.

Proposition 3.1.4. (1) $M_H^{v_0+\alpha}(v_0)$ *is a smooth surface. If* α *is general, then* $\overline{M}_H^{v_0+\alpha}(v_0) = M_H^{v_0+\alpha}(v_0)$ *is projective.*
(2) *If* $\overline{M}_H^{v_0+\alpha}(v_0) = M_H^{v_0+\alpha}(v_0)$*, then it is a K3 surface.*

For the structure of $\overline{M}_H^{v_0}(v_0)$, as in [O-Y], we have the following.

Theorem 3.1.5. *(cf.* [O-Y, Thm. 0.1]*)*
(1) $\overline{M}_H^{v_0}(v_0)$ *is normal and the singular points* $q_1, q_2, \dots, q_m$ *of* $\overline{M}_H^{v_0}(v_0)$ *correspond to the S-equivalence classes of properly* v_0*-twisted semi-stable objects.*
(2) *For a suitable choice of* α *with* $|\langle\alpha^2\rangle| \ll 1$*, there is a surjective morphism* $\pi : \overline{M}_H^{v_0+\alpha}(v_0) = M_H^{v_0+\alpha}(v_0) \to \overline{M}_H^{v_0}(v_0)$ *which becomes a minimal resolution of the singularities.*
(3) *Let* $\bigoplus_{j\ge 0} E_{ij}^{\oplus a'_{ij}}$ *be the S-equivalence class corresponding to* q_i*, where* E_{ij} *are* v_0*-twisted stable objects.*
  (a) *Then the matrix* $(-\langle v(E_{ij}), v(E_{ik})\rangle)_{j,k\ge 0}$ *is of affine type* $\tilde{A}, \tilde{D}, \tilde{E}$*.*
  (b) *Assume that* $a'_{i0} = 1$*. Then the singularity of* $\overline{M}_H^{v_0}(v_0)$ *at* q_i *is a rational double point of type* A, D, E *according as the type of the matrix* $(-\langle v(E_{ij}), v(E_{ik})\rangle)_{j,k\ge 1}$*.*

Remark 3.1.6. A (-2)-vector $u \in L := v_0^\perp \cap \widehat{H}^\perp \cap H^*(X,\mathbb{Z})_{\mathrm{alg}}$ is *numerically irreducible*, if there is no decomposition $u = \sum_i b_i u_i$ such that $u_i \in L$, $\langle u_i^2\rangle = -2$, $\operatorname{rk} u > \operatorname{rk} u_i > 0$, $b_i \in \mathbb{Z}_{>0}$. If u is numerically irreducible, as we shall see in Proposition 3.2.14, there is a v_0-twisted stable object E with $v(E) = u$. In particular, if there is a decomposition $v_0 = \sum_{i\ge 0} a_i u_i$ such that $u_i \in L$ are numerically irreducible, $\langle u_i^2\rangle = -2$, $\operatorname{rk} u_i > 0$ and $a_i \in \mathbb{Z}_{>0}$, then there are v_0-stable objects E_i such that $v(E_i) = u_i$, and hence $v_0 = v(\oplus_i E_i^{\oplus a_i})$. Thus the types of the singularities are determined by the sublattice L of $H^*(X,\mathbb{Z})$.

We shall give a proof of this theorem in subsection 3.2. We assume that $\alpha \in (v_0^\perp \cap \varrho_X^\perp) \otimes \mathbb{Q}$ is general and set $X' := M_H^{v_0+\alpha}(v_0)$. X' is a $K3$ surface. We have a morphism $\phi : X' \to \overline{M}_H^{v_0}(v_0)$. We shall explain some cohomological properties of the Fourier-Mukai transform associated to X'. Let $\mathcal{E}$ be a universal family as a twisted object on $X' \times X$. For simplicity, we assume that $\mathcal{E}$ is an untwisted object on $X' \times X$. But all results hold even if $\mathcal{E}$ is a twisted object. We set

$$
\begin{aligned}
G_1 &:= \mathcal{E}_{|\{x'\}\times X} \in K(X),\\
G_2 &:= \mathcal{E}^\vee_{|X'\times\{x\}} \in K(X'),\\
G_3 &:= \mathcal{E}_{|X'\times\{x\}} \in K(X')
\end{aligned}
\tag{3.2}
$$

for some $x \in X$ and $x' \in X'$. We also set

$$
w_0 := v(\mathcal{E}^\vee_{|X'\times\{x\}}) = r_0 + \widetilde{\xi}_0 + \widetilde{a}_0 \varrho_{X'}, \widetilde{\xi}_0 \in \mathrm{NS}(X'). \tag{3.3}
$$

We set $\Phi^\alpha := \Phi^{\mathcal{E}^\vee}_{X\to X'}$ and $\widehat{\Phi}^\alpha := \Phi^{\mathcal{E}}_{X'\to X}$. Thus

$$
\Phi^\alpha(x) := \mathbf{R}\operatorname{Hom}_{p_{X'}}(\mathcal{E}, p_X^*(x)), x \in \mathbf{D}(X), \tag{3.4}
$$

and $\widehat{\Phi}^\alpha : \mathbf{D}(X') \to \mathbf{D}(X)$ by

$$
\widehat{\Phi}^\alpha(y) := \mathbf{R}\operatorname{Hom}_{p_X}(\mathcal{E}^\vee, p_{X'}^*(y)), y \in \mathbf{D}(X'), \tag{3.5}
$$

where $\operatorname{Hom}_{p_Z}(-,-) = p_{Z*}\mathcal{H}om_{\mathcal{O}_{X'\times X}}(-,-)$, $Z = X, X'$ are the sheaves of relative homomorphisms.

Theorem 3.1.7 ([Br2], [O])**.** Φ^α *is an equivalence of categories and the inverse is given by* $\widehat{\Phi}^\alpha[2]$*.*

For $D \in H^2(X,\mathbb{Q})$, we set

$$
\begin{aligned}
\widehat{D} &:= -\left[\Phi^\alpha\left(D + \frac{(D,\xi_0)}{r_0}\varrho_X\right)\right]_1\\
&= \left[p_{X'*}\left(\left(c_2(\mathcal{E}) - \frac{r_0-1}{2r_0}(c_1(\mathcal{E})^2)\right) \cup p_X^*(D)\right)\right]_1 \in H^2(X',\mathbb{Q}),
\end{aligned}
\tag{3.6}
$$

where $[\ ]_1$ means the projection to $H^2(X',\mathbb{Q})$.

Lemma 3.1.8. *(cf.* [Y5, Lem. 1.4]*)* $r_0\widehat{H}$ *is a nef and big divisor on* X' *which defines a contraction* $\pi' : X' \to Y'$ *of* X' *to a normal surface* Y'*. There is a morphism* $\psi : Y' \to \overline{M}_H^{v_0}(v_0)$ *such that* $\phi = \psi \circ \pi'$*.*

Proof. Let G be a local projective generator of $\mathcal{C}$ such that $\tau(G) = 2\tau(G_1)$ (Lemma 3.1.3). Applying Lemma 1.4.6, we have an ample line bundle $\mathcal{L}(\zeta)$ on $\overline{M}_H^G(v_0) = \overline{M}_H^{v_0}(v_0)$. By the definition of $\widehat{H}$, $c_1(\phi^*(\mathcal{L}(\zeta))) = r_0\widehat{H}$. Hence our claim holds. □

Proposition 3.1.9. *(cf.* [Y5, Prop. 1.5]*)*

(1) *Every element $v \in H^*(X,\mathbb{Z})$ can be uniquely written as*

$$v = lv_0 + a\varrho_X + d\left(H + \frac{1}{r_0}(H,\xi_0)\varrho_X\right) + \left(D + \frac{1}{r_0}(D,\xi_0)\varrho_X\right),$$

where

$$\begin{aligned} l =& \frac{\operatorname{rk} v}{\operatorname{rk} v_0} = -\frac{\langle v, \varrho_X\rangle}{\operatorname{rk} v_0} \in \frac{1}{r_0}\mathbb{Z}, \\ a =& -\frac{\langle v, v_0\rangle}{\operatorname{rk} v_0} \in \frac{1}{r_0}\mathbb{Z}, \\ d =& \frac{\deg_{G_1}(v)}{\operatorname{rk} v_0(H^2)} \in \frac{1}{r_0(H^2)}\mathbb{Z} \end{aligned} \tag{3.7}$$

and $D \in H^2(X,\mathbb{Q}) \cap H^\perp$. Moreover $v \in v(\mathbf{D}(X))$ if and only if $D \in \mathrm{NS}(X) \otimes \mathbb{Q} \cap H^\perp$.

(2)

$$\begin{aligned} &\Phi^\alpha\left(lv_0 + a\varrho_X + \left(dH + D + \frac{1}{r_0}(dH + D, \xi_0)\varrho_X\right)\right) \\ =& l\varrho_{X'} + aw_0 - \left(d\widehat{H} + \widehat{D} + \frac{1}{r_0}(d\widehat{H} + \widehat{D}, \widetilde{\xi}_0)\varrho_{X'}\right) \end{aligned} \tag{3.8}$$

where $D \in H^2(X,\mathbb{Q}) \cap H^\perp$.

(3)

$$\deg_{G_1}(v) = -\deg_{G_2}(\Phi^\alpha(v)).$$

In particular, $\deg_{G_2}(w) \in \mathbb{Z}$ for $w \in H^*(X',\mathbb{Z})$ and

$$\min\{\deg_{G_1}(E) > 0 | E \in K(X)\} = \min\{\deg_{G_2}(F) > 0 | F \in K(X')\}.$$

3.2. **Proof of Theorem 3.1.5.** We shall choose a special α and study the structure of the moduli spaces. We first prove the following. The normalness of $\overline{M}_H^{v_0}(v_0)$ will be proved in Proposition 3.2.13.

Proposition 3.2.1. (1) *$\psi : Y' \to \overline{M}_H^{v_0}(v_0)$ is bijective.*

(2) *The singular points of Y' correspond to properly v_0-twisted semi-stable objects.*

(3) *Let $\bigoplus_{j\ge 0} E_{ij}^{\oplus a'_{ij}}$ be the S-equivalence class of a properly v_0-twisted semi-stable object, where E_{ij} are v_0-twisted stable. Then the matrix $(-\langle v(E_{ij}), v(E_{ik})\rangle)_{j,k\ge 0}$ is of affine type $\tilde{A}, \tilde{D}, \tilde{E}$. We assume that $a_{i0} = 1$. Then $\psi^{-1}(\bigoplus_{j\ge 0} E_{ij}^{\oplus a'_{ij}})$ is a rational double point of type A, D, E according as the type of the matrix $(-\langle v(E_{ij}), v(E_{ik})\rangle)_{j,k\ge 1}$.*

3.2.1. *Proof of Proposition 3.2.1.* We note that $M_H^{v_0}(v_0)$ is smooth and ϕ, ψ are isomorphic over $M_H^{v_0}(v_0)$. Hence the singular points of Y' are in the inverse image of $\overline{M}_H^{v_0}(v_0) \setminus M_H^{v_0}(v_0)$. Thus we may concentrate on the locus of properly v_0-twisted semi-stable objects. The first claim of Proposition 3.2.1 (3) follows from the following.

Lemma 3.2.2. *Assume that E is S-equivalent to $\bigoplus_{j\ge 0} E_{ij}^{\oplus a'_{ij}}$, where E_{ij} are v_0-twisted stable objects. Then the matrix $(-\langle v(E_{ij}), v(E_{ik})\rangle)_{j,k\ge 0}$ is of type $\widetilde{A}, \widetilde{D}, \widetilde{E}$. Moreover $\langle v(E_{ij}), v(E_{kl})\rangle = 0$, if $\bigoplus_{j\ge 0} E_{ij}^{\oplus a'_{ij}} \not\cong \bigoplus_{l\ge 0} E_{kl}^{\oplus a'_{kl}}$.*

Proof. Since $\deg_{G_1}(E) = \chi(G_1, E) = 0$, $\deg_{G_1}(E_{ij}) = \chi(G_1, E_{ij}) = 0$, which implies that $v(E_{ij}) \in v_0^\perp \cap \widehat{H}^\perp$. Since $(v_0^\perp \cap \widehat{H}^\perp)/\mathbb{Z}v_0$ is negative definite, applying Lemma 6.1.1 (1), we see that the matrix is of type $\widetilde{A}, \widetilde{D}, \widetilde{E}$. We note that $\bigoplus_{j\ge 0} E_{ij}^{\oplus a'_{ij}} \not\cong \bigoplus_{l\ge 0} E_{kl}^{\oplus a'_{kl}}$ implies that $\{E_{i0}, E_{i1}, ..., E_{is'_i}\} \ne \{E_{k0}, E_{k1}, ..., E_{ks'_k}\}$. Since $\chi(E_{ij}, E_{kl}) > 0$ implies that $E_{ij} \cong E_{kl}$, $\{v(E_{i0}), v(E_{i1}), ..., v(E_{is'_i})\} \ne \{v(E_{k0}), v(E_{k1}), ..., v(E_{ks'_k})\}$. Then the second claim follows from Lemma 6.1.1 (2). □

By this lemma, we may assume that $a'_{i0} = 1$ for all i. THen we can choose a sufficiently small $\alpha \in v_0^\perp$ such that $-\langle \alpha, v(E_{ij})\rangle > 0$ for all $j > 0$. We have the following.

Lemma 3.2.3. *Lemma 2.2.15 holds, if we replace ϱ_X by v_0 and the α-stability by the $v_0 + \alpha$-twisted stability.*

Proof. (1) Assume that F is S-equivalent to $\bigoplus_{j\ge 0} F_{ij}^{\oplus c_{ij}}$, where F_{ij} are v_0-twisted stable objects. If $v(F) = v(\oplus_{j\ge 0} E_{ij}^{\oplus b_{ij}})$, $b_{i0} = 1$, then applying Lemma 3.2.2 to $\bigoplus_{j\ge 0} F_{ij}^{\oplus c_{ij}} \oplus \bigoplus_{j\ge 0} E_{ij}^{\oplus(a_{ij}-b_{ij})}$ and $\bigoplus_{j\ge 0} E_{ij}^{\oplus a_{ij}}$, we get $\bigoplus_{j\ge 0} F_{ij}^{\oplus c_{ij}} \oplus \bigoplus_{j>0} E_{ij}^{\oplus(a_{ij}-b_{ij})} \cong \bigoplus_{j\ge 0} E_{ij}^{\oplus a_{ij}}$, which implies the claim. Then the proofs of (2), (3) and (4) are the same. □

Lemma 3.2.4. (1) *We set*

$$C'_{ij} := \{x' \in X' | \operatorname{Hom}(\mathcal{E}_{|\{x'\}\times X}, E_{ij}) \neq 0\}, j > 0. \tag{3.9}$$

Then C'_{ij} is a smooth rational curve.

(2)

$$\phi^{-1}(\bigoplus_{j\geq 0} E_{ij}^{\oplus a'_{ij}}) = \{x' \in X' | \operatorname{Hom}(E_{i0}, \mathcal{E}_{|\{x'\}\times X}) \neq 0\} = \cup_j C'_{ij}. \tag{3.10}$$

In particular, ϕ and ψ are surjective.

Proof. The proof is the same as in Lemma 2.2.18. □

We also have the following lemma whose proof is the same as of Lemma 2.3.11.

Lemma 3.2.5. *$\Phi^\alpha(E_{ij})[1]$ is a line bundle on C'_{ij}. In particular, $\langle v(E_{ij}), v(E_{kl})\rangle = (C'_{ij}, C'_{kl})$. We define b'_{ij} by $\Phi^\alpha(E_{ij}) = \mathcal{O}_{C'_{ij}}(b'_{ij})[-1]$.*

This lemma shows that the configuration of $\{C'_{ij} | j > 0\}$ is of type A, D, E. Since $(\widehat{H}, C'_{ij}) = 0$, $\cup_j C'_{ij}$ is contracted to a rational double point of Y'. Hence Proposition 3.2.1 (2) and (3) hold. Since $\psi^{-1}(\bigoplus_{j\geq 0} E_{ij}^{\oplus a'_{ij}})$ is a point, ψ is injective. Thus Proposition 3.2.1 (1) also holds.

We shall prove the normality in Proposition 3.2.13.

3.2.2. *Perverse coherent sheaves on X' and the normality of $\overline{M}_H^{v_0}(v_0)$.* We set $Z'_i := \pi^{-1}(q_i) = \sum_{j=1}^{s'_i} a'_{ij} C'_{ij}$. Then E_{i0} is a subobject of $\mathcal{E}_{|\{x'\}\times X}$ for $x' \in Z'_i$ and we have an exact sequence

$$0 \to E_{i0} \to \mathcal{E}_{|\{x'\}\times X} \to F \to 0,\ x' \in Z'_i \tag{3.11}$$

where F is a v_0-twisted semi-stable object with $\operatorname{gr}(F) = \oplus_{j=1}^{s'_i} E_{ij}^{\oplus a'_{ij}}$. Then we get an exact sequence

$$0 \to \Phi^\alpha(F)[1] \to \Phi^\alpha(E_{i0})[2] \to \mathbb{C}_{x'} \to 0 \tag{3.12}$$

in $\operatorname{Coh}(X')$. Thus WIT_2 holds for E_{i0} with respect to Φ^α.

Definition 3.2.6. We set $A'_{i0} := \Phi^\alpha(E_{i0})[2]$ and $A'_{ij} := \Phi^\alpha(E_{ij})[2] = \mathcal{O}_{C'_{ij}}(b'_{ij})[1]$ for $j > 0$.

Lemma 3.2.7. (1) $\operatorname{Hom}(A'_{i0}, A'_{ij}[-1]) = \operatorname{Ext}^1(A'_{i0}, A'_{ij}[-1]) = 0$.

(2) *We set $\mathbf{b}'_i := (b'_{i1}, b'_{i2}, \dots, b'_{is'_i})$. Then $A'_{i0} \cong A_0(\mathbf{b}'_i)$. In particular,* $\operatorname{Hom}(A'_{i0}, \mathbb{C}_{x'}) = \mathbb{C}$ *for $x' \in Z'_i$.*

(3) *Irreducible objects of* $\operatorname{Per}(X'/Y', \mathbf{b}'_1, ..., \mathbf{b}'_m)$ *are*

$$A'_{ij}\ (1 \leq i \leq m, 0 \leq j \leq s'_i),\ \mathbb{C}_{x'}\ (x' \in X' \setminus \cup_i Z'_i). \tag{3.13}$$

Proof. (1) We have

$$\begin{aligned}\operatorname{Hom}(A'_{i0}, A'_{ij}[k]) &= \operatorname{Hom}(\Phi^\alpha(E_{i0})[2], \Phi^\alpha(E_{ij})[2+k]) \\ &= \operatorname{Hom}(E_{i0}, E_{ij}[k]) = 0\end{aligned} \tag{3.14}$$

for $k = -1, 0$.

(2) By (3.12) and (1), we can apply Lemma 2.1.8 to prove $A'_{i0} = A_0(\mathbf{b}'_i) = A_{q_i}$. (3) is a consequence of (2) and Proposition 1.2.19. □

Definition 3.2.8. We set

$$\begin{aligned}\operatorname{Per}(X'/Y') &:= \operatorname{Per}(X'/Y', \mathbf{b}'_1, \dots, \mathbf{b}'_m), \\ \operatorname{Per}(X'/Y')^D &:= \operatorname{Per}(X'/Y', -\mathbf{b}'_1 + 2\mathbf{b}_0, \dots, -\mathbf{b}'_m + 2\mathbf{b}_0)^*, \mathbf{b}_0 := (-1, -1, ..., -1).\end{aligned} \tag{3.15}$$

Remark 3.2.9. Assume that $\alpha \in v_0^\perp$ satisfies $-\langle v(E_{ij}), \alpha\rangle < 0$, $j > 0$. Then $\Phi(E_{ij})[2] = \mathcal{O}_{C'_{ij}}(b''_{ij})$, $j > 0$ and $\Phi(E_{i0})[2] = A_0(\mathbf{b}''_i)[1]$ belong to $\operatorname{Per}(X'/Y', \mathbf{b}''_1, \dots, \mathbf{b}''_m)^*$, where $\mathbf{b}''_i = (b''_{i0}, ..., b''_{is'_i})$.

Lemma 3.2.10. *There is a local projecive generator G of* $\operatorname{Per}(X'/Y')$ *such that $\tau(G) = 2\tau(G_2)$. Moreover $G^\vee$ is a local projective generator of* $\operatorname{Per}(X'/Y')^D$.

Proof. Since $\chi(G_2, A_{ij}) = \chi(\mathbb{C}_x, E_{ij}) = \operatorname{rk} E_{ij} > 0$, we get our claim by Proposition 2.4.1. The second claim follows from the definition of $\operatorname{Per}(X'/Y')^D$ and Lemma 1.1.8. □

Lemma 3.2.11. *Let E be an object of $\mathcal{C}$ such that E is G_1-twisted stable and $\deg_{G_1}(E) = \chi(G_1, E) = 0$. Then $E \cong E_{ij}$ or $E \cong \mathcal{E}_{|\{x'\}\times X}$, $x' \in X' \setminus \cup_i Z'_i$.*

Proof. Since $\chi(G_1, E) = 0$, there is a point $x' \in X'$ such that $\operatorname{Hom}(\mathcal{E}_{|\{x'\}\times X}, E) \neq 0$ or $\operatorname{Hom}(E, \mathcal{E}_{|\{x'\}\times X}) \neq 0$. Then E is a quotient object or a subobject of $\mathcal{E}_{|\{x'\}\times X}$, which implies the claim. □

Definition 3.2.12. (1) Let $\mathcal{C}_{v_0}$ be the full subcategory of $\mathcal{C}$ generated by E_{ij} and $\mathcal{E}_{|\{x'\}\times X}$, $x' \in X'$. That is $\mathcal{C}_{v_0}$ consists of v_0-twisted semi-stable objects E with $\deg_{G_1}(E) = \chi(G_1, E) = 0$.
(2) Let $\operatorname{Per}(X'/Y')_0$ be the full subcategory of $\operatorname{Per}(X'/Y')$ consisting of 0-dimensional objects.

Proposition 3.2.13. (1) *$\Phi^\alpha[2]$ induces an equivalence $\mathcal{C}_{v_0} \to \operatorname{Per}(X'/Y')_0$.*
(2) *Moreover $\Phi^\alpha[2]$ induces an isomorphism $\mathcal{M}_H^{v_0+\beta}(v_0)^{ss} \cong \mathcal{M}_{\widehat{H}}^{G,\Phi^\alpha(\beta)}(\varrho_{X'})^{ss}$, where $\beta \in (v_0^\perp \cap \varrho_X^\perp)\otimes\mathbb{Q}$ is sufficiently small and G an arbitrary projective generator of $\operatorname{Per}(X'/Y')$.*
(3) *$\overline{M}_H^{v_0+\beta}(v_0) \cong \overline{M}_{\widehat{H}}^{G,\Phi^\alpha(\beta)}(\varrho_{X'})$. In particular, $\overline{M}_H^{v_0}(v_0)$ is a normal surface.*

Proof. (1) We note that $\Phi^\alpha(E_{ij})[2] = A'_{ij}$ and $\Phi^\alpha(\mathcal{E}_{|\{x'\}\times X})[2] = \mathbb{C}_{x'}$, $x' \in X'$. Hence the claim holds. (2) We note that $E \in \mathcal{M}_H^{v_0}(v_0)^{ss}$ is $v_0+\beta$-twisted semi-stable, if $\chi(\beta, F) = \chi(v_0+\beta, F) \leq 0$ for all subsheaf F of E with $\deg_{G_1}(F) = \chi(G_1, F) = 0$. Since $\chi(\Phi^\alpha(\beta), \Phi^\alpha(F)) = \chi(\beta, F)$, $\Phi^\alpha(E)[2]$ is $(G_2, \Phi^\alpha(\beta))$-twisted semi-stable. Then Remark 1.5.5 implies that $\Phi^\alpha(E)[2]$ is $(G, \Phi^\alpha(\beta))$-twisted semi-stable for any G. The first claim of (3) follows from (2). In the notation of subsection 2.2, $\overline{M}_{\widehat{H}}^{G,0}(\varrho_{X'}) \cong (X')^0$. Hence the second claim of (3) follows from Proposition 2.2.10. □

Proposition 3.2.14. *Let $u \in H^{ev}(X, \mathbb{Z})_{\text{alg}}$ be a Mukai vector such that $u \in v_0^\perp \cap \widehat{H}^\perp$, $0 < \operatorname{rk} u < \operatorname{rk} v_0$ and $\langle u^2 \rangle = -2$. Then $u = \sum_j b_j v(E_{ij})$, $0 \leq b_j \leq a_{ij}$. In particular, $\overline{M}_H^{v_0}(u) \neq \emptyset$.*

Proof. Since $u \in v_0^\perp \cap \widehat{H}^\perp$, $\Phi^\alpha(u) = (0, D, b)$, $D \in \operatorname{NS}(X')$, $b \in \mathbb{Z}$ and $(D, \widehat{H}) = 0$. Since $(D^2) = -2$, D or $-D$ is an effective divisor supported on an exceptional locus Z'_i. Hence $\Phi^\alpha(u) \in \oplus_{j=0}^{s'_i}\mathbb{Z}\Phi^\alpha(E_{ij}) = \oplus_{j=1}^{s'_i}\mathbb{Z}C_{ij} \oplus \mathbb{Z}\varrho_X$. By the basic properties of the root systems of affine Lie algebra, $\Phi^\alpha(u) = c\Phi^\alpha(v_0) \pm \sum_{j>0} c_j \Phi^\alpha(E_{ij})$, $0 \leq c_j \leq a_{ij}$. Then $\operatorname{rk} u = cr \pm \sum_{j>0} c_j \operatorname{rk} E_{ij}$. Since $\sum_{j>0} c_j \operatorname{rk} E_{ij} \leq \sum_{j>0} a_{ij} \operatorname{rk} E_{ij} < r$, we get $u = \sum_{j>0} c_j v(E_{ij})$ or $u = v_0 - \sum_{j>0} c_j v(E_{ij})$. Therefore the claim holds. □

3.3. **Walls and chambers for the moduli spaces of dimension 2.** We shall study the dependence of $\overline{M}_H^w(v_0)$ on w. We set

$$\begin{array}{rccc} \delta: & \operatorname{NS}(X)\otimes\mathbb{Q} & \to & H^*(X,\mathbb{Q}) \\ & D & \mapsto & D + \frac{(D,\xi_0)}{r_0}\varrho_X. \end{array} \tag{3.16}$$

We may assume that $w = v_0 + \alpha, \alpha \in \delta(H^\perp)$ (cf. [O-Y, sect. 1.1]). We set

$$\mathcal{U} := \left\{ u \in v(\mathbf{D}(X)) \,\middle|\, \begin{array}{l} \langle u^2 \rangle = -2, \langle v_0, u \rangle \leq 0, \langle \delta(H), u \rangle = 0, \\ 0 < \operatorname{rk} u < \operatorname{rk} v_0 \end{array} \right\}. \tag{3.17}$$

For a fixed v_0 and $H, \mathcal{U}$ is a finite set. For $u \in \mathcal{U}$, we define a wall $W_u \subset \delta(H^\perp)\otimes_{\mathbb{Q}}\mathbb{R}$ with respect to v by

$$W_u := \{\alpha \in \delta(H^\perp)\otimes\mathbb{R} | \ \langle v_0 + \alpha, u \rangle = 0\}. \tag{3.18}$$

A connected component of $\delta(H^\perp)\otimes_{\mathbb{Q}}\mathbb{R} \setminus \cup_{u\in\mathcal{U}} W_u$ is said to be a chamber.

Lemma 3.3.1. *If α does not lie on any wall W_u, $u \in \mathcal{U}$, then $\overline{M}_H^{v_0+\alpha}(v_0) = M_H^{v_0+\alpha}(v_0)$. In particular, $\overline{M}_H^{v_0+\alpha}(v_0)$ is a K3 surface.*

We are interested in the $v_0+\alpha$-twisted stability with a sufficiently small $|\langle \alpha^2 \rangle|$. So we may assume that

$$u \in \mathcal{U}' := \{u \in \mathcal{U} | \langle v_0, u \rangle = 0\}. \tag{3.19}$$

For an $\alpha \in \delta(H^\perp)$ with $|\langle \alpha^2 \rangle| \ll 1$, let F be a $v_0+\alpha$-twisted stable torsion free object such that
(i) $\langle v(F)^2 \rangle = -2$,
(ii) $\langle v(F), \delta(H) \rangle / \operatorname{rk} F = (c_1(F), H)/\operatorname{rk} F - (\xi_0, H)/r_0 = 0$ and
(iii) $\langle v_0, v(F) \rangle = \langle \alpha, v(F) \rangle = 0$.

By (i), F is a rigid torsion free object.

Proposition 3.3.2. *(*[O-Y, Prop. 1.12]*) We set $\alpha^\pm := \pm\epsilon v(F) + \alpha$, where $0 < \epsilon \ll 1$. Then T_F induces an isomorphism*

$$\begin{array}{ccc} \mathcal{M}_H^{v+\alpha^-}(v)^{ss} & \to & \mathcal{M}_H^{v+\alpha^+}(v)^{ss} \\ E & \mapsto & T_F(E) \end{array} \tag{3.20}$$

which preserves the S-equivalence classes. Hence we have an isomorphism

$$\overline{M}_H^{v+\alpha^-}(v) \to \overline{M}_H^{v+\alpha^+}(v). \tag{3.21}$$

Remark 3.3.3. In [O-Y], we considered the functor $T_F[-1]$.

Combining Proposition 3.3.2 with Lemma 2.3.20, we get the following Corollary.

Corollary 3.3.4.

$$\Phi_{X'\to X}^{\mathcal{E}^{v_0+\alpha^+}} \cong T_F \circ \Phi_{X'\to X}^{\mathcal{E}^{v_0+\alpha^-}} \cong \Phi_{X'\to X}^{\mathcal{E}^{v_0+\alpha^-}} \circ T_A, \tag{3.22}$$

where $A := \Phi_{X\to X'}^{(\mathcal{E}^{v_0+\alpha^-})^\vee[2]}(F)$.

Assume that $\mathcal{E}^{v_0+\alpha}_{|\{x'\}\times X}$ is S-equivalent to $\oplus_i E_i'^{\oplus a_i'}$. Then $\alpha \in (\sum_i \mathbb{Q}v(E_i'))^\perp$.

Remark 3.3.5. If α belongs to exactly one wall W_u, $u \in \mathcal{U}$, then there is a $v+\alpha$-twisted stable object F with $v(F) = u$. So we can apply Propositions 3.3.2. Moreover $A = \mathcal{O}_C(b)$, where C is a smooth rational curve defined by

$$C := \{x' \in X' | \operatorname{Ext}^2(\mathcal{E}^{v_0+\alpha^-}_{|\{x'\}\times X}, F) \neq 0\}. \tag{3.23}$$

Proposition 3.3.6. *Let G be an object of $\mathbf{D}(X)$ such that $\chi(G, E_{ij}) > 0$ for all i,j and*

$$\operatorname{Hom}(G, E_{ij}[k]) = \operatorname{Hom}(G, E[k]) = 0, k \neq 2 \tag{3.24}$$

for all $E \in M_H^{G_1}(v_0)$ and i,j. Assume that $\alpha \in \delta(H^\perp) \setminus \cup_{u\in\mathcal{U}'} W_u$ is sufficiently small.

(1) *$G^\alpha := \Phi^\alpha(G)$ is a locally free sheaf on X' and $\mathcal{A}' := \pi_*((G^\alpha)^\vee \otimes G^\alpha)$ is a reflexive sheaf on Y' which is independent of the choice of α.*
(2) *$\mathbf{R}\pi_*((G^\alpha)^\vee \otimes \_) \circ \Phi^\alpha : \mathbf{D}(X) \to \mathbf{D}_{\mathcal{A}'}(Y')$ is independent of the choice of α.*

Proof. We take a small $\alpha \in \delta(H^\perp)$ with $-\langle \alpha, v(E_{ij})\rangle > 0$, $j > 0$. By the base change theorem, G^α is a locally free sheaf on X'. Let A'_{ij} be objects of $\operatorname{Per}(X'/Y')$ in subsection 3.2. Then we have $\operatorname{Hom}(G^\alpha, A'_{ij}[k]) = 0$ for $k \neq 0$ and $\operatorname{Hom}(G^\alpha, A'_{ij}) \neq 0$. Assume that $\alpha' \in \delta(H^\perp)$ belongs to another chamber. We set $X'' := M_H^{v_0+\alpha'}(v_0)$. By Proposition 3.2.13 (2), $X'' \cong M_{\widehat{H}}^{G^\alpha, \Phi^\alpha(\alpha')}(\varrho_{X'})$ and $\mathcal{F} := \Phi_{X\to X'}^{(\mathcal{E}^\alpha)^\vee[2]}(\mathcal{E}^{\alpha'})$ is the universal family of $\Phi^\alpha(\alpha')$-twisted stable objects, where $\mathcal{E}^{\alpha'}$ is the universal family associated to α'. We have $\Phi^{\alpha'} = \Phi_{X'\to X''}^{\mathcal{F}^\vee[2]} \circ \Phi^\alpha$. In particular, $G^{\alpha'} = \Phi_{X'\to X''}^{\mathcal{F}^\vee[2]}(G^\alpha)$. Then the claim follows from Proposition 2.3.4. □

3.4. **A tilting appeared in** [Br4] **and its generalizations.** From now on, we assume that α satisfies $-\langle \alpha, v(E_{ij})\rangle > 0$ for all $j > 0$ and set

$$\Phi := \Phi^\alpha,\ \widehat{\Phi} := \widehat{\Phi}^\alpha. \tag{3.25}$$

By Proposition 3.3.6, the assumption is not essential.

Definition 3.4.1. We set

$$\mathfrak{C}_i := \begin{cases} \mathcal{C}, & i = 1, \\ \operatorname{Per}(X'/Y'), & i = 2, \\ \operatorname{Per}(X'/Y')^D, & i = 3. \end{cases} \tag{3.26}$$

For an object $E \in \mathfrak{C}_i$, we define the G_i-twisted Hilbert polynomial by

$$\chi(G_i, E(n)) := \sum_j (-1)^j \dim \operatorname{Hom}(G_i, E(n)[j]), \tag{3.27}$$

where $E(n) := E(nH)$, $i = 1$ and $E(n) := E(n\widehat{H})$, $i = 2, 3$.

Then Lemma 3.1.3 and Lemma 3.2.10 imply the following.

Lemma 3.4.2. *$\chi(G_i, E(n)) > 0$ for $E \neq 0$ and $n \gg 0$, that is, (i) $\operatorname{rk} E > 0$ or (ii) $\operatorname{rk} E = 0, \deg_{G_i}(E) > 0$ or (iii) $\operatorname{rk} E = \deg_{G_i}(E) = 0, \chi(G_i, E) > 0$.*

Definition 3.4.3. Let $E \neq 0$ be an object of $\mathfrak{C}_i$.

(1) There is a (unique) filtration

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E \tag{3.28}$$

such that each $E_j := F_j/F_{j-1}$ is a torsion object or a torsion free G-twisted semi-stable object and

$$(\operatorname{rk} E_{j+1})\chi(G_i, E_j(n)) > (\operatorname{rk} E_j)\chi(G_i, E_{j+1}(n)), n \gg 0. \tag{3.29}$$

We call it the *Harder-Narasimhan filtration* of E.

(2) In the notation of (1), we set

$$\mu_{\max,G_i}(E) := \begin{cases} \mu_{G_i}(E_1), & \operatorname{rk} E_1 > 0 \\ \infty, & \operatorname{rk} E_1 = 0, \end{cases}$$
$$\mu_{\min,G_i}(E) := \begin{cases} \mu_{G_i}(E_s), & \operatorname{rk} E_s > 0 \\ \infty, & \operatorname{rk} E_s = 0. \end{cases} \tag{3.30}$$

Remark 3.4.4. An object $E \neq 0$ has a torsion if and only if $\mu_{\max,G_i}(E) = \infty$ and E is a torsion object if and only if $\mu_{\min,G_i}(E) = \infty$.

We define several torsion pairs of $\mathfrak{C}_i$.

Definition 3.4.5. (1) Let $\mathfrak{T}_i^\mu$ (resp. $\overline{\mathfrak{T}}_i^\mu$) be the full subcategory of $\mathfrak{C}_i$ such that $E \in \mathfrak{C}_i$ belongs to $\mathfrak{T}_i^\mu$ (resp. $\overline{\mathfrak{T}}_i^\mu$) if (i) E is a torsion object or (ii) $\mu_{\min,G_i}(E) > 0$ (resp. $\mu_{\min,G_i}(E) \geq 0$).
(2) Let $\mathfrak{F}_i^\mu$ (resp. $\overline{\mathfrak{F}}_i^\mu$) be the full subcategory of $\mathfrak{C}_i$ such that $E \in \mathfrak{C}_i$ belongs to $\mathfrak{T}_i^\mu$ (resp. $\overline{\mathfrak{F}}_i^\mu$) if $E = 0$ or E is a torsion free object with $\mu_{\max,G_i}(E) \leq 0$ (resp. $\mu_{\max,G_i}(E) < 0$).

Definition 3.4.6. (1) Let $\mathfrak{T}_i$ (resp. $\overline{\mathfrak{T}}_i$) be the full subcategory of $\mathfrak{C}_i$ such that $E \in \mathfrak{C}_i$ belongs to $\mathfrak{T}_i$ (resp. $\overline{\mathfrak{T}}_i$) if (i) E is a torsion object or (ii) for the Harder-Narasimhan filtration (3.28) of E, E_s satisfies $\mu_{G_i}(E_s) > 0$ or $\mu_{G_i}(E_s) = 0$ and $\chi(G_i, E_s) > 0$ (resp. $\mu_{G_i}(E_s) = 0$ and $\chi(G_i, E_s) \geq 0$).
(2) Let $\mathfrak{F}_i$ (resp. $\overline{\mathfrak{F}}_i$) be the full subcategory of $\mathfrak{C}_i$ such that $E \in \mathfrak{C}_i$ belongs to $\mathfrak{F}_i$ (resp. $\overline{\mathfrak{F}}_i$) if E is a torsion free object and for the Harder-Narasimhan filtration (3.28) of E, E_1 satisfies $\mu_{G_i}(E_1) < 0$ or $\mu_{G_i}(E_1) = 0$ and $\chi(G_i, E_1) \leq 0$ (resp. $\mu_{G_i}(E_1) = 0$ and $\chi(G_i, E_1) < 0$).

Definition 3.4.7. $(\mathfrak{T}_i^\mu, \mathfrak{F}_i^\mu)$, $(\overline{\mathfrak{T}}_i^\mu, \overline{\mathfrak{F}}_i^\mu)$, $(\mathfrak{T}_i, \mathfrak{F}_i)$ and $(\overline{\mathfrak{T}}_i, \overline{\mathfrak{F}}_i)$ are torsion pairs of $\mathfrak{C}_i$. We denote the tiltings of $\mathfrak{C}_i$ by $\mathfrak{A}_i^\mu$, $\overline{\mathfrak{A}}_i^\mu$, $\mathfrak{A}_i$ and $\overline{\mathfrak{A}}_i$ respectively.

We note that $\mathfrak{T}_1^\mu \subset \mathfrak{T}_1$. We shall study the condition $\mathfrak{T}_1^\mu = \mathfrak{T}_1$. We start with the following lemma.

Lemma 3.4.8. *Let E be a local projective generator of $\mathfrak{C}_i$. Then* $\operatorname{Ext}^1(E, F) = 0$ *for all 0-dimensional objects F of $\mathfrak{C}_i$. In particular, if E is a subobject of a torsion free object E' such that E'/E is 0-dimensional, then $E' = E$.*

Proof. We only treat the case where $i = 1$. Then $\mathbf{R}\pi_*(E^\vee \otimes F) = \pi_*(E^\vee \otimes F)$ is a 0-dimensional sheaf on Y. Hence we get $\operatorname{Ext}^1(E, F) = H^1(Y, \pi_*(E^\vee \otimes F)) = 0$. □

Lemma 3.4.9. *Assume that $\mathcal{E}_{|\{x'\}\times X}$ is a μ-stable local projective generator of $\mathcal{C}$ for a general $x' \in X'$.*
(1) $\mathfrak{T}_1 = \mathfrak{T}_1^\mu$.
(2) *Every μ-semi-stable object $E \in \mathcal{C}$ with $\deg_{G_1}(E) = \chi(G_1, E) = 0$ is G_1-twisted semi-stable. Moreover if E is G_1-twisted stable, then it is μ-stable.*
(3) *Let E be a μ-semi-stable object $E \in \mathcal{C}$ with* $\operatorname{rk} E > 0$, $\deg_{G_1}(E) = \chi(G_1, E) = 0$. *Then* $\operatorname{Ext}^i(E, S) = 0$, $i \neq 0$ *for any irreducible object $S \in \mathcal{C}$.*
(4) *$\mathcal{E}_{|\{x'\}\times X}$ is a local projective generator of $\mathcal{C}$ for any $x' \in X'$.*

Proof. (1) Let E be a μ-stable object of $\mathcal{C}$ with $\deg_{G_1}(E) = 0$ and $\chi(G_1, E) > 0$. Since $\operatorname{Hom}(E, \mathcal{E}_{|\{x'\}\times X}) = 0$ for all $x' \in X'$, $\operatorname{Hom}(\mathcal{E}_{|\{x'\}\times X}, E) \neq 0$ for all $x' \in X'$. Assume that $\mathcal{E}_{|\{x'\}\times X}$ is a μ-stable local projective generator. By Lemma 3.4.8 and $\operatorname{Hom}(\mathcal{E}_{|\{x'\}\times X}, E) \neq 0$, we get $E \cong \mathcal{E}_{|\{x'\}\times X}$. Therefore $\chi(G_1, E) \leq 0$ for all μ-stable object $E \in \mathcal{C}$ with $\deg_{G_1}(E) = 0$. Hence we get $\mathfrak{T}_1 = \mathfrak{T}_1^\mu$.

(2) Let E' be a subobject of E with $\deg_{G_1}(E) = 0$. Then (1) implies that $\chi(G_1, E') \leq 0$. Hence E is G_1-twisted semi-stable. If E/E' is torsion free, then we also have $\chi(G_1, E/E') \leq 0$, which implies that $\chi(G_1, E') = \chi(G_1, E/E') = 0$. Thus E is properly G_1-twisted semi-stable. Therefore the second claim also holds.

(3) If $\operatorname{Ext}^1(S, E) = \operatorname{Ext}^1(E, S)^\vee \neq 0$, then a non-trivial extension

$$0 \to E \to E' \to S \to 0 \tag{3.31}$$

gives a μ-semi-stable object E' with $\chi(G_1, E') = \chi(G_1, S) > 0$. On the other hand, (1) implies that $\chi(G_1, E') \leq 0$. Therefore $\operatorname{Ext}^1(E, S) = 0$. Since S is a torsion object, $\operatorname{Ext}^2(E, S) \cong \operatorname{Hom}(S, E)^\vee = 0$.

(4) Since $\mathcal{E}_{|\{x'\}\times X}$ is a μ-semi-stable object with $\deg_{G_1}(\mathcal{E}_{|\{x'\}\times X}) = \chi(G_1, \mathcal{E}_{|\{x'\}\times X}) = 0$, $\mathcal{E}_{|\{x'\}\times X} \in \mathcal{C}$ and satisfies the assertion of (3). By Lemma 3.4.2, $\chi(\mathcal{E}_{|\{x'\}\times X}, S) = \chi(G_1, S) > 0$ for any irreducible object S. Then $\mathcal{E}_{|\{x'\}\times X}$ is locally free and is a local projective generator by Proposition 1.1.22. □

Remark 3.4.10. By the proof of Lemma 3.4.9, $\mathcal{E}_{|\{x'\}\times X}$, $x' \in X'$ is a local projective generator of $\mathcal{C}$ if $\mathfrak{T}_1 = \mathfrak{T}_1^\mu$. Indeed if $\mathfrak{T}_1 = \mathfrak{T}_1^\mu$, then the same proofs of (2), (3) and (4) work.

3.5. **Equivalence between $\mathfrak{A}_1$ and $\mathfrak{A}_2^\mu$.**

Lemma 3.5.1. (1) *If $E \in \mathfrak{T}_1$, then $\operatorname{Hom}(E, E_{ij}) = \operatorname{Hom}(E, \mathcal{E}_{|\{x'\}\times X}) = 0$ for all i, j and $x' \in X'$. In particular, $H^2(\Phi(E)) = 0$.*

(2) *If $E \in \mathfrak{F}_1$, then $\operatorname{Hom}(\mathcal{E}_{|\{x'\}\times X}, E) = 0$ for a general $x' \in X'$. In particular, $H^0(\Phi(E)) = 0$.*

Proof. (1) The first claim is obvious. The second claim is a consequence of the Serre duality and the base change theorem (see the proof of Lemma 3.5.2 (2)).

(2) If there is a non-zero morphism $\phi : \mathcal{E}_{|\{x'\}\times X} \to E$, we see that ϕ is injective and $\operatorname{coker}\phi \in \mathfrak{F}_1$. By the induction on $\operatorname{rk} E$, we get the first claim. The second claim follows by the base change theorem. □

Lemma 3.5.2. *Let E be an object of $\mathcal{C}$.*

(1) *Assume that $\operatorname{Hom}(E_{ij}, E[q]) = \operatorname{Hom}(\mathcal{E}_{|\{x'\}\times X}, E[q]) = 0$ for all i, j, $x' \in X'$ and $q > 0$. Then $\Phi(E) \in \operatorname{Per}(X'/Y')$.*

(2) *There is a complex*

$$0 \to W_0 \to W_1 \to W_2 \to 0 \tag{3.32}$$

such that W_i are local projective objects of $\operatorname{Per}(X'/Y')$ and $\Phi(E)$ is quasi-isomorphic to this complex.

(3) *$H^0({}^pH^2(\Phi(E))) = H^2(\Phi(E))$ and ${}^pH^0(\Phi(E)) \subset H^0(\Phi(E))$. In particular, ${}^pH^0(\Phi(E))$ is torsion free.*

(4) *If $\operatorname{Hom}(E, E_{ij}) = 0$ for all i, j and $\operatorname{Hom}(E, \mathcal{E}_{|\{x'\}\times X}) = 0$ for all $x' \in X'$, then ${}^pH^2(\Phi(E)) = 0$. In particular, if $E \in \mathfrak{T}_1$, then ${}^pH^2(\Phi(E)) = 0$.*

(5) *If $E \in \mathfrak{F}_1$, then ${}^pH^0(\Phi(E)) = 0$.*

Proof. (1) We note that $F \in \operatorname{Per}(X'/Y')$ is 0 if and only if $\operatorname{Hom}(F, A'_{ij}) = \operatorname{Hom}(F, A'_{i0}) = \operatorname{Hom}(F, \mathbb{C}_{x'}) = 0$ for all i, $j > 0$ and $x' \in X'$. Since

$$\begin{aligned} \operatorname{Hom}(\Phi(E)[q], \Phi(E_{ij})[2]) &\cong \operatorname{Hom}(E[q], E_{ij}[2]) \cong \operatorname{Hom}(E_{ij}, E[q])^\vee, \\ \operatorname{Hom}(\Phi(E)[q], \Phi(\mathcal{E}_{|\{x'\}\times X})[2]) &\cong \operatorname{Hom}(E[q], \mathcal{E}_{|\{x'\}\times X}[2]) \cong \operatorname{Hom}(\mathcal{E}_{|\{x'\}\times X}, E[q])^\vee, \end{aligned} \tag{3.33}$$

we have ${}^pH^q(\Phi(E)) = 0$ for $q > 0$, which implies that $\Phi(E) \in \operatorname{Per}(X'/Y')$. Thus the claim (1) holds.

(2)

We take a resolution of E

$$0 \to V_{-2} \to V_{-1} \to V_0 \to E \to 0 \tag{3.34}$$

such that $V_{-k} = G(-n_k)^{\oplus N_k}$, $n_k \gg 0$ for $k = 0, 1$, where G is a local projective generator of $\mathcal{C}$. By using the Serre duality, our choice of n_k implies that $\operatorname{Hom}(\mathcal{E}_{|\{x'\}\times X}, V_{-k}[q]) = \operatorname{Hom}(E_{ij}, V_{-k}[q]) = 0$ for $q \neq 2$ and $k = 0, 1$. Then we also have $\operatorname{Hom}(\mathcal{E}_{|\{x'\}\times X}, V_{-2}[q]) = \operatorname{Hom}(E_{ij}, V_{-2}[q]) = 0$ for $q \neq 2$. Hence $\Phi(V_{-k})[2]$, $k = 0, 1, 2$ are locally free sheaves on X'. Since $\operatorname{Hom}(\Phi(V_{-k})[2], A'_{ij}[q]) = \operatorname{Hom}(\Phi(V_{-k})[2], \Phi(E_{ij})[2+q]) = \operatorname{Hom}(V_{-k}, E_{ij}[q]) = 0$, $q > 0$, $W_{2-k} := \Phi(V_{-k})[2]$, $k = 0, 1, 2$ are local projective objects of $\operatorname{Per}(X'/Y')$ and the associated complex $W_\bullet$ defines the required complex.

(3) is obvious. (4) follows from the proof of (1) and Lemma 3.5.1 (1). (5) follows from (3) and Lemma 3.5.1 (2). □

Definition 3.5.3. (1) We set $\Phi^i(E) := {}^pH^i(\Phi(E)) \in \operatorname{Per}(X'/Y')$ and $\widehat{\Phi}^i(E) := {}^pH^i(\widehat{\Phi}(E)) \in \mathcal{C}$.

(2) We say that WIT_i holds for $E \in \mathcal{C}$ (resp. $F \in \operatorname{Per}(X'/Y')$) with respect to Φ (resp. $\widehat{\Phi}$), if $\Phi^j(E) = 0$ (resp. $\widehat{\Phi}^j(F)) = 0$) for $j \neq i$.

Lemma 3.5.4. *Let E be an object of $\mathcal{C}$.*

(1) *If WIT_0 holds for E with respect to Φ, then $E \in \mathfrak{T}_1$.*

(2) *If WIT_2 holds for E with respect to Φ, then $E \in \mathfrak{F}_1$. In particular, E is torsion free. Moreover if $\Phi^2(E)$ does not contain a 0-dimensional object, then $E \in \overline{\mathfrak{F}}_1^\mu$.*

Proof. For an object $E \in \mathcal{C}$, there is an exact sequence

$$0 \to E_1 \to E \to E_2 \to 0 \tag{3.35}$$

such that $E_1 \in \mathfrak{T}_1$ and $E_2 \in \mathfrak{F}_1$. Applying Φ to this exact sequence, we get a long exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Phi^0(E_1) & \longrightarrow & \Phi^0(E) & \longrightarrow & \Phi^0(E_2) \\ & \longrightarrow & \Phi^1(E_1) & \longrightarrow & \Phi^1(E) & \longrightarrow & \Phi^1(E_2) \\ & \longrightarrow & \Phi^2(E_1) & \longrightarrow & \Phi^2(E) & \longrightarrow & \Phi^2(E_2) \longrightarrow 0. \end{array} \tag{3.36}$$

By Lemma 3.5.2 (4),(5), $\Phi^0(E_2) = \Phi^2(E_1) = 0$. If WIT_0 holds for E, then we get $\Phi(E_2) = 0$. Hence (1) holds. If WIT_2 holds for E, then we get $\Phi(E_1) = 0$. Thus the first part of (2) holds. Assume that there is an exact sequence

$$0 \to E_2' \to E \to E_2'' \to 0 \tag{3.37}$$

such that E_2' is a μ-semi-stable object with $\deg_{G_1}(E_2') = 0$ and $E_2'' \in \overline{\mathfrak{F}}_1^\mu$. By the first part of (2), we get $\chi(G_1, E_2') \leq 0$. By Lemma 3.5.1 (2), $\Phi^0(E_2'') = 0$. Then we see that WIT_2 holds for E_2' and $\deg_{G_2}(\Phi^2(E_2')) = \deg_{G_1}(E_2') = 0$. Since $\operatorname{rk}\Phi^2(E_2') = \chi(G_1, E_2') \leq 0$, $\Phi^2(E_2')$ is a 0-dimensional object. By our assumption, we get that $\Phi^1(E_2'') \to \Phi^2(E_2')$ is an isomorphism. By Lemma 6.3.1 in the appendix, we have $\widehat{\Phi}^0(\Phi^1(E_2'')) = 0$, which implies that $E_2' \cong \widehat{\Phi}^0(\Phi^2(E_2')) = 0$. □

Lemma 3.5.5. *For an object $E \in \mathcal{C}$, $\deg_{G_2}(\Phi^0(E)) \leq 0$ and $\deg_{G_2}(\Phi^2(E)) \geq 0$.*

Proof. We note that

$$\widehat{\Phi}(\Phi^0(E)) = \widehat{\Phi}^2(\Phi^0(E))[-2],\ \widehat{\Phi}(\Phi^2(E)) = \widehat{\Phi}^0(\Phi^2(E)) \tag{3.38}$$

and

$$\deg_{G_2}(\Phi^0(E)) = -\deg_{G_1}(\widehat{\Phi}^2(\Phi^0(E))),\ \deg_{G_2}(\Phi^2(E)) = -\deg_{G_1}(\widehat{\Phi}^0(\Phi^2(E))). \tag{3.39}$$

Since $\widehat{\Phi}^2(\Phi^0(E))$ satisfies WIT_0 with respect to Φ, $\widehat{\Phi}^2(\Phi^0(E)) \in \mathfrak{T}_1$, which implies that $\deg_{G_1}(\widehat{\Phi}^2(\Phi^0(E))) \geq 0$. Since $\widehat{\Phi}^0(\Phi^2(E))$ satisfies WIT_2 with respect to Φ, $\widehat{\Phi}^0(\Phi^2(E)) \in \mathfrak{F}_1$, which implies that $\deg_{G_1}(\widehat{\Phi}^0(\Phi^2(E))) \leq 0$. Therefore our claims hold. □

Lemma 3.5.6. (1) *If $F \in \mathfrak{T}_2^\mu$, then $\widehat{\Phi}^2(F) = 0$.*
(2) *If WIT_0 holds for $F \in \mathrm{Per}(X'/Y')$ with respect to $\widehat{\Phi}$, then $F \in \mathfrak{T}_2^\mu$.*
(3) *If $F \in \mathfrak{F}_2^\mu$, then $\widehat{\Phi}^0(F) = 0$.*
(4) *If WIT_2 holds for $F \in \mathrm{Per}(X'/Y')$ with respect to $\widehat{\Phi}$, then $F \in \mathfrak{F}_2^\mu$.*

Proof. (1) By Lemma 6.3.1 in the appendix, we have an exact sequence

$$F \to \Phi^0(\widehat{\Phi}^2(F)) \xrightarrow{\phi} \Phi^2(\widehat{\Phi}^1(F)) \to 0. \tag{3.40}$$

By Lemma 3.5.5, $\deg_{G_2}(\ker\phi) \leq 0$. Since $\Phi^0(\widehat{\Phi}^2(F))$ is torsion free, $\ker\phi$ is also torsion free. By our assumption of F, we have $\ker\phi = 0$. Then $\Phi^0(\widehat{\Phi}^2(F)) \cong \Phi^2(\widehat{\Phi}^1(F))$ satisfies WIT_0 and WIT_2, which implies that $\Phi^0(\widehat{\Phi}^2(F)) \cong \Phi^2(\widehat{\Phi}^1(F)) \cong 0$. Therefore $\widehat{\Phi}^2(F) = 0$.

(2) Assume that there is an exact sequence

$$0 \to F_1 \to F \to F_2 \to 0 \tag{3.41}$$

such that $F_1 \in \mathfrak{T}_2^\mu$ and $F_2 \in \mathfrak{F}_2^\mu$. By (1), we have $\widehat{\Phi}^2(F_1) = 0$. By a similar exact sequence to (3.36), we see that WIT_0 holds for F_2 and $\deg_{G_1}(\widehat{\Phi}^0(F_2)) = -\deg_{G_2}(F_2) \geq 0$. On the other hand, since WIT_2 holds for $\widehat{\Phi}^0(F_2)$, Lemma 3.5.4 implies that $\widehat{\Phi}^0(F_2) \in \mathfrak{F}_1$. Hence $\deg_{G_1}(\widehat{\Phi}^0(F_2)) = 0$ and $\chi(G_1, \widehat{\Phi}^0(F_2)) \leq 0$. Since $\chi(G_1, \widehat{\Phi}^0(F_2)) = \operatorname{rk} F_2$, we have $\operatorname{rk} F_2 = 0$. Since $\mathfrak{F}_2^\mu$ contains no torsion object except 0, we conclude that $F_2 = 0$.

(3) By Lemma 6.3.1, we have an exact sequence

$$0 \to \Phi^0(\widehat{\Phi}^1(F)) \xrightarrow{\psi} \Phi^2(\widehat{\Phi}^0(F)) \to F. \tag{3.42}$$

By (2), $\Phi^2(\widehat{\Phi}^0(F)) \in \mathfrak{T}_2^\mu$, which implies that $\operatorname{coker}\psi = 0$. Then $\Phi^0(\widehat{\Phi}^1(F)) \cong \Phi^2(\widehat{\Phi}^0(F))$ satisfies WIT_0 and WIT_2, which implies that $\Phi^0(\widehat{\Phi}^1(F)) \cong \Phi^2(\widehat{\Phi}^0(F)) \cong 0$. Therefore $\widehat{\Phi}^0(F) = 0$.

(4) Assume that there is an exact sequence

$$0 \to F_1 \to F \to F_2 \to 0 \tag{3.43}$$

such that $0 \neq F_1 \in \mathfrak{T}_2^\mu$ and $F_2 \in \mathfrak{F}_2^\mu$. By (3), $\widehat{\Phi}^0(F_2) = 0$. By a similar exact sequence to (3.36), we see that WIT_2 holds for F_1 and $\deg_{G_1}(\widehat{\Phi}^2(F_1)) = -\deg_{G_2}(F_1) \leq 0$. Moreover if $\operatorname{rk} F_1 > 0$, then $\deg_{G_1}(\widehat{\Phi}^2(F_1)) < 0$. On the other hand, since WIT_0 holds for $\widehat{\Phi}^2(F_1)$, Lemma 3.5.4 implies that $\widehat{\Phi}^2(F_1) \in \mathfrak{T}_1$. Hence $\operatorname{rk} F_1 = 0$ and $\deg_{G_1}(\widehat{\Phi}^2(F_1)) = 0$. Then $\widehat{\Phi}^2(F_1) \in \mathfrak{T}_1$ implies that $0 < \chi(G_1, \widehat{\Phi}^2(F_1)) = \operatorname{rk} F_1$, which is a contradiction. Therefore $F_1 = 0$. □

Lemma 3.5.7. (1) *Assume that $E \in \mathfrak{T}_1$. Then*
(a) *$\Phi^0(E) \in \mathfrak{F}_2^\mu$.*
(b) *$\Phi^1(E) \in \mathfrak{T}_2^\mu$.*
(c) *$\Phi^2(E) = 0$.*
(2) *Assume that $E \in \mathfrak{F}_1$. Then*

(a) $\Phi^0(E) = 0$.
(b) $\Phi^1(E) \in \mathfrak{F}_2^\mu$.
(c) $\Phi^2(E) \in \mathfrak{T}_2^\mu$.

Proof. We take a decomposition

$$0 \to F_1 \to \Phi^1(E) \to F_2 \to 0 \tag{3.44}$$

with $F_1 \in \mathfrak{T}_2^\mu$ and $F_2 \in \mathfrak{F}_2^\mu$. Applying $\widehat{\Phi}$, we have an exact sequence

$$\begin{aligned} 0 \longrightarrow \widehat{\Phi}^0(F_1) &\longrightarrow \widehat{\Phi}^0(\Phi^1(E)) \longrightarrow \widehat{\Phi}^0(F_2) \\ \longrightarrow \widehat{\Phi}^1(F_1) &\longrightarrow \widehat{\Phi}^1(\Phi^1(E)) \longrightarrow \widehat{\Phi}^1(F_2) \\ \longrightarrow \widehat{\Phi}^2(F_1) &\longrightarrow \widehat{\Phi}^2(\Phi^1(E)) \longrightarrow \widehat{\Phi}^2(F_2) \longrightarrow 0. \end{aligned} \tag{3.45}$$

By Lemma 3.5.6, we have $\widehat{\Phi}^0(F_2) = \widehat{\Phi}^2(F_1) = 0$.

(1) Assume that $E \in \mathfrak{T}_1$. Then (a) follows from Lemma 3.5.6 (4), and (c) follows from Lemma 3.5.2 (4). We prove (b). We assume that $F_2 \neq 0$. By Lemma 6.3.1 and (c), we have $\widehat{\Phi}^2(\Phi^1(E)) = 0$. Then WIT_1 holds for F_2 and $\deg_{G_1}(\widehat{\Phi}^1(F_2)) = \deg_{G_2}(F_2) \leq 0$. By Lemma 6.3.1, we have a surjective homomorphism

$$E \to \widehat{\Phi}^1(\Phi^1(E)). \tag{3.46}$$

Hence $\widehat{\Phi}^1(F_2)$ is a quotient object of E. Since $E \in \mathfrak{T}_1$, we see that $\deg_{G_1}(\widehat{\Phi}^1(F_2)) \geq 0$. Hence $\deg_{G_1}(\widehat{\Phi}^1(F_2)) = 0$. If $\operatorname{rk} \widehat{\Phi}^1(F_2) > 0$, then since $\chi(G_1, \widehat{\Phi}^1(F_2)) = -\operatorname{rk} F_2 < 0$, we get $E \notin \mathfrak{T}_1$. Hence $\operatorname{rk} \widehat{\Phi}^1(F_2) = 0$. Then $\chi(G_1, \widehat{\Phi}^1(F_2)) = -\operatorname{rk} F_2 < 0$ implies that the G_1-twisted Hilbert polynomial of $\widehat{\Phi}^1(F_2)$ is not positive. By Lemma 3.4.2, this is impossible. Therefore $F_2 = 0$.

(2) Assume that $E \in \mathfrak{F}_1$. By Lemma 3.5.2 and Lemma 3.5.6, (a) and (c) hold. We prove (b). Assume that $F_1 \neq 0$. By $\Phi^0(E) = 0$ and Lemma 6.3.1, we have $\widehat{\Phi}^0(\Phi^1(E)) = 0$. Then WIT_1 holds for F_1 and we have an injective morphism $\widehat{\Phi}^1(F_1) \to \widehat{\Phi}^1(\Phi^1(E)) \to E$. Assume that $\dim F_1 \geq 1$. Since $\deg_{G_1}(\widehat{\Phi}^1(F_1)) = \deg_{G_2}(F_1) > 0$, this is impossible. Assume that $\dim F_1 = 0$. Then $\chi(G_2, F_1) > 0$, which implies that $\operatorname{rk} \widehat{\Phi}^1(F_1) = -\chi(G_2, F_1) < 0$. This is a contradiction. Therefore $F_1 = 0$. □

The following is a generalization of a result in [H] (see Remark 3.5.9 below).

Theorem 3.5.8. *Φ induces an equivalence $\mathfrak{A}_1 \to \mathfrak{A}_2^\mu[-1]$. Moreover $\widehat{\Phi}^0(F) \in \overline{\mathfrak{F}}_1^\mu$ if $F \in \mathfrak{T}_2^\mu$ does not contain a 0-dimensional object.*

Proof. For $E \in \mathfrak{A}_1$, we have an exact sequence in $\mathfrak{A}_1$

$$0 \to H^{-1}(E)[1] \to E \to H^0(E) \to 0. \tag{3.47}$$

Then we have an exact triangle

$$\Phi(H^{-1}(E))[2] \to \Phi(E[1]) \to \Phi(H^0(E))[1] \to \Phi(H^{-1}(E))[3]. \tag{3.48}$$

Hence $\Phi^i(E[1]) = 0$ for $i \neq -1, 0$ and we have an exact sequence

$$\begin{aligned} 0 \longrightarrow \Phi^1(H^{-1}(E)) &\longrightarrow \Phi^{-1}(E[1]) \longrightarrow \Phi^0(H^0(E)) \\ \longrightarrow \Phi^2(H^{-1}(E)) &\longrightarrow \Phi^0(E[1]) \longrightarrow \Phi^1(H^0(E)) \longrightarrow 0. \end{aligned} \tag{3.49}$$

By Lemme 3.5.7, $\Phi^{-1}(E[1]) \in \mathfrak{F}_2^\mu$ and $\Phi^0(E[1]) \in \mathfrak{T}_2^\mu$. Therefore $\Phi(E[1]) \in \mathfrak{A}_2^\mu$.

Conversely for $F \in \mathfrak{A}_2^\mu$ and $E_1 \in \mathfrak{A}_1$, $\Phi(E_1)[1] \in \mathfrak{A}_2^\mu$ implies that

$$\begin{aligned} \operatorname{Hom}(\widehat{\Phi}(F)[1], E_1[p]) &= \operatorname{Hom}(F, (\Phi(E_1)[1])[p]) = 0, \ p < 0, \\ \operatorname{Hom}(E_1[p], \widehat{\Phi}(F)[1]) &= \operatorname{Hom}((\Phi(E_1)[1])[p], F) = 0, \ p > 0. \end{aligned} \tag{3.50}$$

Hence $\widehat{\Phi}(F)[1] \in \mathfrak{A}_1$. Therefore the first claim holds.

For the last claim, we note that there is an exact sequence

$$0 \to \Phi^0(\widehat{\Phi}^1(F)) \to \Phi^2(\widehat{\Phi}^0(F)) \to F \tag{3.51}$$

by Lemma 6.3.1. By Lemma 3.5.2 (3), $\Phi^0(\widehat{\Phi}^1(F))$ is torsion free. Hence $\Phi^2(\widehat{\Phi}^0(F))$ does not contain a 0-dimensional object. Then Lemma 3.5.4 (2) implies the claim. □

Remark 3.5.9. In [Y5], we gave a different proof of [H, Prop. 4.2]. Since we used different notations in [Y5], we explain the correspendence of the terminologies: Φ corresponds to $\mathcal{F}_{\mathcal{E}}$ in [Y5], $\mathfrak{A}_2^\mu$ corresponds to $\mathfrak{A}_1$ in [Y5, Thm. 2.1] and $\mathfrak{A}_1$ corresponds to $\mathfrak{A}_2$ or $\mathfrak{A}_2'$ in [Y5, Thm. 2.1, Prop. 2.7].

3.6. **Fourier-Mukai duality for a $K3$ surface.** In this subsection, we shall prove a kind of duality property between (X, H) and $(X', \widehat{H})$. In other words, we show that X is the moduli space of some objects on X' and H is the natural determinant line bundle on the moduli space.

Theorem 3.6.1. *Assume that $\mathbb{C}_x$ is β-stable for all $x \in X$.*

(1) $\mathcal{E}_{|X' \times \{x\}} \in \mathrm{Per}(X'/Y')^D$ *is $G_3 - \Phi(\beta)^\vee$-twisted stable for all $x \in X$ and we have an isomorphism $\phi : X \to M_{\widehat{H}}^{G_3-\Phi(\beta)^\vee}(w_0^\vee)$ by sending $x \in X$ to $\mathcal{E}_{|X' \times \{x\}} \in M_{\widehat{H}}^{G_3-\Phi(\beta)^\vee}(w_0^\vee)$. Moreover we have $H = \widehat{(\widehat{H})}$ under this isomorphism.*

(2) *Assume that $\mathcal{E}_{|\{x'\} \times X}$ is a μ-stable local projective generator of $\mathcal{C}$ for a general $x' \in X'$. Then $\mathcal{E}_{|X' \times \{x\}}$ is a μ-stable local projective generator of $\mathrm{Per}(X'/Y')^D$ for $x \in X \setminus \cup_i Z_i$.*

The proof is similar to that in [Y5, Thm. 2.2]. In particular, if $\mathcal{E}_{|\{x'\} \times X}$ is a μ-stable locally free sheaf for a general $x' \in X'$, then the same proof in [Y5] works. However if $\mathcal{E}_{|\{x'\} \times X}$ is not a μ-stable locally free sheaf for any $x' \in X'$, then we need to introduce a (contravariant) Fourier-Mukai transforms and study their properties. We set

$$
\begin{aligned}
\Psi(E) &:= \mathbf{R}\operatorname{Hom}_{p_{X'}}(p_X^*(E), \mathcal{E}) = \Phi(E)^\vee[-2],\ E \in \mathbf{D}(X),\\
\widehat{\Psi}(F) &:= \mathbf{R}\operatorname{Hom}_{p_X}(p_{X'}^*(F), \mathcal{E}),\ F \in \mathbf{D}(X').
\end{aligned} \tag{3.52}
$$

We shall first study the properties of Ψ and $\widehat{\Psi}$ which are similar to those of Φ and $\widehat{\Phi}$.

We set

$$
\begin{aligned}
\Psi(E_{ij})[2] &= B'_{ij},\ j > 0\\
\Psi(E_{i0})[2] &= B'_{i0}.
\end{aligned} \tag{3.53}
$$

Then the following claims follow from Definition 3.2.8 and Lemma 3.2.7.

Lemma 3.6.2. (1) $B'_{ij} = \mathcal{O}_{C'_{ij}}(-b'_{ij}-2) \in \mathrm{Per}(X'/Y')^D$ *and* $B'_{i0} = A_0(-\mathbf{b}'+2\mathbf{b}_0)^*[1] \in \mathrm{Per}(X'/Y')^D$.

(2) *Irreducible objects of* $\mathrm{Per}(X'/Y')^D$ *are*

$$
B'_{ij}\ (1 \le i \le m, 0 \le j \le s'_i),\ \mathbb{C}_{x'} (x' \in X \setminus \cup_i Z'_i). \tag{3.54}
$$

Lemma 3.6.3. (1) *Assume that $E \in \overline{\mathfrak{T}}_1$. Then $\operatorname{Hom}(E, \mathcal{E}_{|\{x'\} \times X}) = 0$ for a general $x' \in X'$.*

(2) *Assume that $E \in \overline{\mathfrak{F}}_1$. Then $\operatorname{Hom}(\mathcal{E}_{|\{x'\} \times X}, E) = 0$ for all $x' \in X'$.*

Proof. We only prove (1). Let E be a G_1-twisted stable object of $\mathcal{C}$. If $\deg_{G_1}(E) > 0$ or $\deg_{G_1}(E) = 0$ and $\chi(G_1, E) > 0$, then $\operatorname{Hom}(E, \mathcal{E}_{|\{x'\} \times X}) = 0$ for all $x' \in X'$. Assume that $\deg_{G_1}(E) = 0$ and $\chi(G_1, E) = 0$. Then a non-zero homomorphism $E \to \mathcal{E}_{|\{x'\} \times X}$ is an isomorphism if $x' \notin \cup_i Z'_i$. Therefore $\operatorname{Hom}(E, \mathcal{E}_{|\{x'\} \times X}) = 0$ for a general $x' \in X'$. □

Lemma 3.6.4. *Let E be an object of $\mathcal{C}$.*

(1) ${}^pH^i(\Psi(E)) = 0$ *for $i \ge 3$.*

(2) $H^0({}^pH^2(\Psi(E))) = H^2(\Psi(E))$.

(3) ${}^pH^0(\Psi(E)) \subset H^0(\Psi(E))$. *In particular, ${}^pH^0(\Psi(E))$ is torsion free.*

(4) *If $\operatorname{Hom}(E, E_{ij}[2]) = 0$ for all i, j and $\operatorname{Hom}(E, \mathcal{E}_{|\{x'\} \times X}[2]) = 0$ for all $x' \in X'$, then ${}^pH^2(\Psi(E)) = 0$. In particular, if $E \in \overline{\mathfrak{F}}_1$, then ${}^pH^2(\Psi(E)) = 0$.*

(5) *If E satisfies $E \in \overline{\mathfrak{T}}_1$, then ${}^pH^0(\Psi(E)) = 0$.*

Proof. Let $W_\bullet$ be the complex in Lemma 3.5.2 (2). By Remark 1.1.9, $W_i^\vee$ are local projective objects of $\mathrm{Per}(X'/Y')^D$. Since $\Psi(E)$ is represented by the complex $W_\bullet^\vee[-2]$, (1), (2) and (3) follow.

By Lemma 3.6.2, $F \in \mathrm{Per}(X'/Y')^D$ is 0 if and only if $\operatorname{Hom}(F, B'_{ij}) = \operatorname{Hom}(F, \mathbb{C}_{x'}) = 0$ for all i, j and $x' \in X'$.

Since

$$
\begin{aligned}
\operatorname{Hom}(E, E_{ij}[2-p])^\vee &\cong \operatorname{Hom}(\Psi(E)[2-p], \Psi(E_{ij})[2]),\\
\operatorname{Hom}(E, \mathcal{E}_{|\{x'\} \times X}[2-p])^\vee &\cong \operatorname{Hom}(\Psi(E)[2-p], \Psi(\mathcal{E}_{|\{x'\} \times X})[2]),
\end{aligned} \tag{3.55}
$$

we have (4). (5) follows from (3) and Lemma 3.6.3 (1). □

Definition 3.6.5. We set $\Psi^i(E) := {}^pH^i(\Psi(E)) \in \mathrm{Per}(X'/Y')^D$ and $\widehat{\Psi}^i(E) := {}^pH^i(\widehat{\Psi}(E)) \in \mathcal{C}$.

Lemma 3.6.6. *Let E be an object of $\mathcal{C}$.*

(1) *If WIT_0 holds for E with respect to Ψ, then $E \in \overline{\mathfrak{F}}_1$.*

(2) *If WIT_2 holds for E with respect to Ψ, then $E \in \overline{\mathfrak{T}}_1$. If $\Psi^2(E)$ does not contain a 0-dimensional object, then $E \in \mathfrak{T}_1$.*

Proof. For an object E of $\mathcal{C}$, there is an exact sequence

$$0 \to E_1 \to E \to E_2 \to 0 \tag{3.56}$$

such that $E_1 \in \overline{\mathfrak{T}}_1$ and $E_2 \in \overline{\mathfrak{F}}_1$. Applying Ψ to this exact sequence, we get a long exact sequence

$$\begin{aligned} 0 \longrightarrow \Psi^0(E_2) \longrightarrow \Psi^0(E) \longrightarrow \Psi^0(E_1) \\ \longrightarrow \Psi^1(E_2) \longrightarrow \Psi^1(E) \longrightarrow \Psi^1(E_1) \\ \longrightarrow \Psi^2(E_2) \longrightarrow \Psi^2(E) \longrightarrow \Psi^2(E_1) \longrightarrow 0 \end{aligned} \tag{3.57}$$

By Lemma 3.6.4, we have $\Psi^0(E_1) = \Psi^2(E_2) = 0$. If WIT_0 holds for E, then we get $\Psi(E_1) = 0$. Hence (1) holds. If WIT_2 holds for E, then we get $\Psi(E_2) = 0$. Thus the first part of (2) holds. Assume that $\Psi^2(E)$ does not have a non-zero 0-dimensional subobject. We take a decomposition

$$0 \to E_1 \to E \to E_2 \to 0 \tag{3.58}$$

such that $E_1 \in \mathfrak{T}_1$ and E_2 is a G_1-twisted semi-stable object with $\deg_{G_1}(E_2) = \chi(G_1, E_2) = 0$. Then $\Psi^0(E_1) = \Psi^0(E_2) = \Psi^1(E_2) = 0$. In particular, WIT_2 holds for E_2 with respect to Ψ. Then $\Psi^2(E_2)$ is a torsion object with $\deg_{G_3}(\Psi^2(E_2)) = 0$, which implies that $\Psi^2(E_2)$ is 0-dimensional. Our assumption implies that $\Psi^1(E_1) \cong \Psi^2(E_2)$. By Lemma 6.3.2 and $\widehat{\Psi}^0(\Psi^0(E_1)) = 0$, we get $E_2 = \widehat{\Psi}^2(\Psi^2(E_2)) = \widehat{\Psi}^2(\Psi^1(E_1)) = 0$. □

Lemma 3.6.7. *Let E be a μ-semi-stable object with $\deg_{G_1}(E) = 0$. If WIT_0 holds for E, then $E = 0$.*

Proof. If WIT_0 holds for $E \neq 0$, then $\chi(G_1, E) = \operatorname{rk} \Psi(E) \geq 0$. On the other hand, Lemma 3.6.6 implies that $\chi(G_1, E) < 0$. Therefore $E = 0$. □

Lemma 3.6.8. *If WIT_0 holds for E with respect to Ψ, then $E \in \overline{\mathfrak{F}}_1^{\mu}$.*

Proof. Assume that there is an exact sequence

$$0 \to E_1 \to E \to E_2 \to 0 \tag{3.59}$$

such that E_1 is a μ-semi-stable object with $\deg_{G_1}(E_1) = 0$ and $E_2 \in \overline{\mathfrak{F}}_1^{\mu}$. Then we have $\Psi^2(E_2) = 0$. By the exact sequence (3.57), WIT_0 holds for E_1. Then Lemma 3.6.7 implies that $E_1 = 0$. □

Lemma 3.6.9. *If $E \in \overline{\mathfrak{T}}_1^{\mu}$, then $\Psi^0(E) = 0$.*

Proof. We may assume that E is a μ-semi-stable object or a torsion object. If $\deg_{G_1}(E) > 0$, then the claim holds by the base change theorem. Assume that $\deg_{G_1}(E) = 0$. By Lemma 6.3.2, we have an exact sequence

$$E \to \widehat{\Psi}^0(\Psi^0(E)) \to \widehat{\Psi}^2(\Psi^1(E)) \to 0. \tag{3.60}$$

By Lemma 3.6.8, $\widehat{\Psi}^0(\Psi^0(E)) \in \overline{\mathfrak{F}}_1^{\mu}$. Since E is a μ-semi-stable object with $\deg_{G_1}(E) = 0$, $E \to \widehat{\Psi}^0(\Psi^0(E))$ is a zero map. Then $\widehat{\Psi}^0(\Psi^0(E)) \cong \widehat{\Psi}^2(\Psi^1(E))$ satisfies WIT_0 and WIT_2, which implies that $\widehat{\Psi}^0(\Psi^0(E)) \cong \widehat{\Psi}^2(\Psi^1(E)) \cong 0$. Therefore $\Psi^0(E) = 0$. □

Lemma 3.6.10.

$$\deg_{G_3}(\Psi^0(E)) \leq 0,\ \deg_{G_3}(\Psi^2(E)) \geq 0. \tag{3.61}$$

Proof. We note that

$$\deg_{G_3}(\Psi^i(E)) = \deg_{G_1}(\widehat{\Psi}^i(\Psi^i(E))) \tag{3.62}$$

for $i = 0, 2$ by Lemma 6.3.2. Then the claim follows from Lemma 3.6.6. □

Proof of Theorem 3.6.1.
(1) We first prove the G_3-twisted semi-stability of $\mathcal{E}_{|X' \times \{x\}}$ for all $x \in X$. It is sufficient to prove the following lemma.

Lemma 3.6.11. *Let E be a 0-dimensional object of $\mathcal{C}$. Then WIT_2 holds for E with respect to Ψ and $\Psi^2(E)$ is a G_3-twisted semi-stable object such that $\deg_{G_3}(\Psi^2(E)) = \chi(G_3, \Psi^2(E)) = 0$. Moreover if E is irreducible, then $\Psi^2(E)$ is G_3-twisted stable.*

Proof. We first prove that E satisfies WIT_2 with respect to Ψ. We may assume that E is irreducible. Then we get $\operatorname{Hom}(E, \mathcal{E}_{|\{x'\} \times X}) = 0$ for all x'. Hence $\Psi^0(E) = 0$. We shall prove that $\Psi^1(E) = 0$ by showing $\widehat{\Psi}^i(\Psi^1(E)) = 0$ for $i = 0, 1, 2$. By Lemma 6.3.2, $\widehat{\Psi}^2(\Psi^1(E)) = 0$ and we have an exact sequence

$$0 \to \widehat{\Psi}^0(\Psi^1(E)) \to \widehat{\Psi}^2(\Psi^2(E)) \to E \to \widehat{\Psi}^1(\Psi^1(E)) \to 0. \tag{3.63}$$

By Lemma 3.6.6 and Lemma 6.3.2, $\widehat{\Psi}^0(\Psi^1(E)) \in \overline{\mathfrak{F}}_1$ and $\widehat{\Psi}^2(\Psi^2(E)) \in \overline{\mathfrak{T}}_1$. Since E is 0-dimensional, $\widehat{\Psi}^0(\Psi^1(E))$ is μ-semi-stable and $\deg_{G_1}(\widehat{\Psi}^0(\Psi^1(E))) = \deg_{G_1}(\widehat{\Psi}^2(\Psi^2(E))) = 0$. By Lemma 3.6.7, $\widehat{\Psi}^0(\Psi^1(E)) =$

0. Since E is an irreducible object, $\widehat{\Psi}^2(\Psi^2(E))=0$ or $\widehat{\Psi}^1(\Psi^1(E))=0$. If $\widehat{\Psi}^2(\Psi^2(E))=0$, then $\Psi^2(E)=0$. Since $\chi(G_1,E)>0$, we get a contradiction. Hence we also have $\widehat{\Psi}^1(\Psi^1(E))=0$, which implies that $\Psi^1(E)=0$. Therefore WIT_2 holds for E with respect to Ψ.

We next prove that $\Psi^2(E)$ is G_3-twisted semi-stable. Assume that there is an exact sequence

$$0\to F_1\to\Psi^2(E)\to F_2\to 0 \tag{3.64}$$

such that $F_1\in\mathrm{Per}(X'/Y')^D$, $\deg_{G_3}(F_1)\geq 0$ and $F_2\in\mathrm{Per}(X'/Y')^D$. Applying $\widehat{\Psi}$ to this exact sequence, we get a long exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \widehat{\Psi}^0(F_2) & \longrightarrow & 0 & \longrightarrow & \widehat{\Psi}^0(F_1) \\ & \longrightarrow & \widehat{\Psi}^1(F_2) & \longrightarrow & 0 & \longrightarrow & \widehat{\Psi}^1(F_1) \\ & \longrightarrow & \widehat{\Psi}^2(F_2) & \longrightarrow & E & \longrightarrow & \widehat{\Psi}^2(F_1) \longrightarrow 0. \end{array} \tag{3.65}$$

By Lemma 6.3.2, WIT_2 holds for $\widehat{\Psi}^2(F_2)$. Hence $\widehat{\Psi}^2(F_2)\in\overline{\mathfrak{T}}_1$, in particular, we have $\deg_{G_1}(\widehat{\Psi}^2(F_2))\geq 0$. By Lemma 6.3.2, WIT_0 holds for $\widehat{\Psi}^1(F_2)\cong\widehat{\Psi}^0(F_1)$. Hence $\widehat{\Psi}^1(F_2)\in\overline{\mathfrak{F}}_1$, which implies that $\deg_{G_1}(\widehat{\Psi}^1(F_2))\leq 0$. Therefore $\deg_{G_1}(\widehat{\Psi}(F_2))\geq 0$. On the other hand, $\deg_{G_1}(\widehat{\Psi}(F_2))=\deg_{G_3}(F_2)\leq 0$. Hence $\widehat{\Psi}^1(F_2)$ is a μ-semi-stable object with $\deg_{G_1}(\widehat{\Psi}^1(F_2))=0$ and $\deg_{G_3}(F_2)=0$. Then Lemma 3.6.7 implies that $\widehat{\Psi}^1(F_2)=0$. If $\chi(G_3,F_2)\leq 0$, then $\mathrm{rk}\,\widehat{\Psi}^2(F_2)=\chi(G_3,F_2)$ implies that $\chi(G_3,F_2)=0$ and $\widehat{\Psi}^2(F_2)$ is a torsion object. This in particular means that $\Psi^2(E)$ is G_2-twisted semi-stable. We further assume that E is irreducible. Since $\deg_{G_1}(\widehat{\Psi}^2(F_2))=0$, $\widehat{\Psi}^2(F_2)$ is a 0-dimensional object. Then WIT_2 holds for $\widehat{\Psi}^1(F_1)$, $\widehat{\Psi}^2(F_1)$ and $\widehat{\Psi}^2(F_2)$ with respect to Ψ. Since $\Psi^2(\widehat{\Psi}^1(F_1))=0$, $\widehat{\Psi}^1(F_1)=0$. Then $\widehat{\Psi}^2(F_2)=0$ or $\widehat{\Psi}^2(F_1)=0$, which implies that $F_1=0$ or $F_2=0$. Therefore $\Psi^2(E)$ is G_3-twisted stable. □

We continue the proof of (1). Assume that there is an exact sequence in $\mathrm{Per}(X'/Y')^D$

$$0\to F_1\to\mathcal{E}_{|X'\times\{x\}}\to F_2\to 0 \tag{3.66}$$

such that $\deg_{G_3}(F_1)=\chi(G_3,F_1)=0$. By the proof of Lemma 3.6.11, WIT_2 holds for F_1 and F_2. Thus we get an exact sequence

$$0\to\widehat{\Psi}^2(F_2)\to\mathbb{C}_x\to\widehat{\Psi}^2(F_1)\to 0 \tag{3.67}$$

Since $\mathbb{C}_x$ is β-stable, $\chi(\beta,\widehat{\Psi}^2(F_2))<0$, which implies that $\chi(-\Psi(\beta),F_2)>0$. Therefore $\mathcal{E}_{|X'\times\{x\}}$ is $G_3-\Psi(\beta)$-twisted stable. Then we have an injective morphism $\phi:X\to\overline{M}_{\widehat{H}}^{G_3+\alpha'}(w_0^\vee)$ by sending $x\in X$ to $\mathcal{E}_{|X'\times\{x\}}$, where $\alpha'=-\Psi(\beta)$. By a standard argument, we see that ϕ is an isomorphism. We note that $[\widehat{\Psi}(\widehat{H}+(\widehat{H},\widetilde{\xi}_0)/r_0\varrho_{X'})]_1$ is the pull-back of the canonical polarization on $\overline{M}_{\widehat{H}}^{G_3}(w_0^\vee)$. Hence under the identification $M_{\widehat{H}}^{G_3+\alpha'}(w_0^\vee)\cong X$, $\widehat{(\widehat{H})}=H$.

(2) Assume that $\mathcal{E}_{|\{x'\}\times X}$ is a μ-stable local projective generator for a general $x'\in X'$. By Lemma 3.6.13 (2) below, we only need to prove the μ-stability of $\mathcal{E}_{|X'\times\{x\}}$ for $x\in X\setminus\cup_i Z_i$. We shall study the exact sequence (3.64) in Lemma 3.6.11, where $E=\mathbb{C}_x$. We may assume that F_2 satisfies $\deg_{G_3}(F_2)=0$ and $\chi(G_3,F_2)>0$. Then WIT_2 holds for F_2 by the proof of Lemma 3.6.11. We shall first prove that $\widehat{\Psi}^1(F_1)$ does not contain a 0-dimensional object. Let T_1 be the 0-dimensional subobject of $\widehat{\Psi}^1(F_1)$. Then we have a surjective morphism $\Psi^2(\widehat{\Psi}^1(F_1))\to\Psi^2(T_1)$. Since WIT_2 holds for T_1 with respect to Ψ and $\Psi^0(\widehat{\Psi}^0(F_1))\to\Psi^2(\widehat{\Psi}^1(F_1))$ is surjective, we get $T_1=0$. By Lemma 3.6.6, $\widehat{\Psi}^2(F_2)\in\overline{\mathfrak{T}}_1$. Then Lemma 3.4.9 and $\deg_{G_1}(\widehat{\Psi}^2(F_2))=0$ imply that $\widehat{\Psi}^2(F_2)$ is an extension of a G_1-semi-stable object E_1 with $\deg_{G_1}(E_1)=\chi(G_1,E_1)=0$ by a 0-dimensional object T. Since $T\cap\widehat{\Psi}^1(F_1)=0$, $T=\mathbb{C}_x$ or 0. By our assumption, $\Psi^2(E_1)$ is a torsion object. By the exact sequence

$$\Psi^2(E_1)\to F_2\to\Psi^2(T)\to 0, \tag{3.68}$$

we have $\mathrm{rk}\,F_2=(\mathrm{rk}\,\mathcal{E}_{|X'\times\{x\}})\dim T$, which implies that $\mathrm{rk}\,F_2=\mathrm{rk}\,\mathcal{E}_{|X'\times\{x\}}$ or $\mathrm{rk}\,F_2=0$. Therefore $\mathcal{E}_{|X'\times\{x\}}$ is μ-stable. □

Lemma 3.6.12. *If $\mathcal{E}_{|\{x'\}\times X}, x'\in X'$ and E_{ij} are locally free on an open subset X^0 of X, then $\mathcal{E}_{|X'\times\{x\}}$ is a local projective generator of $\mathrm{Per}(X'/Y')^D$ for $x\in X^0$.*

Proof. We first note that $\mathcal{E}_{|X'\times\{x\}}\in\mathrm{Coh}(X')$ by Theorem 3.6.1. The claim follows from the following equalities:

$$\begin{aligned} \mathrm{Hom}(\mathcal{E}_{|X'\times\{x\}},\mathbb{C}_{x'}[k]) &= \mathrm{Hom}(\Psi(\mathbb{C}_x),\Psi(\mathcal{E}_{|\{x'\}\times X})[k])=\mathrm{Hom}(\mathcal{E}_{|\{x'\}\times X},\mathbb{C}_x[k])=0,\\ \mathrm{Hom}(\mathcal{E}_{|X'\times\{x\}},B'_{ij}[k]) &= \mathrm{Hom}(\Psi(\mathbb{C}_x),\Psi(E_{ij})[k])=\mathrm{Hom}(E_{ij},\mathbb{C}_x[k])=0 \end{aligned} \tag{3.69}$$

for $x\in X^0$, $x'\in X'$ and $k\neq 0$. □

Lemma 3.6.13. (1) *If $X=Y$ and Y' is not smooth, then $\mathcal{E}_{|X'\times\{x\}}$ is a local projective generator of $\operatorname{Per}(X'/Y')^D$ for all $x\in X$.*

(2) *If $\mathcal{E}_{|\{x'\}\times X}$ is a μ-stable local projective object of $\mathcal{C}$ for a general $x'\in X'$, then $\mathcal{E}_{|X'\times\{x\}}$ is a local projective generator of $\operatorname{Per}(X'/Y')^D$ for all $x\in X$.*

Proof. (1) We first note that $E_{ij}\in\operatorname{Coh}(X)=\mathcal{C}$ are locally free sheaves for all i,j. Assume that $E:=\mathcal{E}_{|\{x'\}\times X}$ is not locally free for a point $x'\in X'$. Then we have a morphism from an open subscheme Q of $\operatorname{Quot}^n_{E^{\vee\vee}/X/\mathbb{C}}$ to X', where $n=\dim(E^{\vee\vee}/E)$. Since $\dim X'=2$, this morphism is dominant. Hence $\mathcal{E}_{|\{x'\}\times X}$ is non-locally free for all $x'\in X'$. Since $\mathcal{E}_{|\{x'\}\times X}$ is locally free if x' belongs to the exceptional locus, $\mathcal{E}_{|\{x'\}\times X}$ is locally free for any $x'\in X'$. Then the claim follows from Lemma 3.6.12.

(2) The claim follows from Lemma 3.4.9 (3), (4) and the proof of Lemma 3.6.12. □

In the remaining of this subsection, we shall prove the following result.

Proposition 3.6.14. *$\Psi:\mathbf{D}(X)\to\mathbf{D}(X')_{op}$ induces an equivalence $\overline{\mathfrak{A}}_1^{\mu}[-2]\to(\overline{\mathfrak{A}}_3)_{op}$.*

We first note that the following two lemmas hold thanks to Theorem 3.6.1.

Lemma 3.6.15 (cf. Lem. 3.6.3)**.** (1) *Assume that $F\in\overline{\mathfrak{T}}_3$. Then $\operatorname{Hom}(F,\mathcal{E}_{|X'\times\{x\}})=0$ for a general $x\in X$. In particular, $\widehat{\Psi}^0(F)=0$.*

(2) *Assume that $F\in\overline{\mathfrak{F}}_3$. Then $\operatorname{Hom}(\mathcal{E}_{|X'\times\{x\}},F)=0$ for all $x\in X$. In particular, $\widehat{\Psi}^2(F)=0$.*

Lemma 3.6.16 (cf. Lem. 3.6.6, Lem. 3.6.8)**.** *Let F be an object of $\operatorname{Per}(X'/Y')^D$.*

(1) *If WIT_0 holds for F with respect to $\widehat{\Psi}$, then $F\in\overline{\mathfrak{F}}_3^{\mu}(\subset\overline{\mathfrak{F}}_3)$.*

(2) *If WIT_2 holds for F with respect to $\widehat{\Psi}$, then $F\in\overline{\mathfrak{T}}_3$. If $\widehat{\Psi}^2(F)$ does not contain a 0-dimensional subobject, then $F\in\mathfrak{T}_3$.*

Lemma 3.6.17. (1) *Assume that $E\in\overline{\mathfrak{T}}_1^{\mu}$. Then*

(a) $\Psi^0(E)=0$.

(b) $\Psi^1(E)\in\overline{\mathfrak{F}}_3$.

(c) $\Psi^2(E)\in\overline{\mathfrak{T}}_3$. *Moreover if E does not contain a non-trivial 0-dimensional subobject, then $\Psi^2(E)\in\mathfrak{T}_3$.*

(2) *Assume that $E\in\overline{\mathfrak{F}}_1^{\mu}$. Then*

(a) $\Psi^0(E)\in\overline{\mathfrak{F}}_3$.

(b) $\Psi^1(E)\in\overline{\mathfrak{T}}_3$.

(c) $\Psi^2(E)=0$.

Proof. We take a decomposition

$$0\to F_1\to\Psi^1(E)\to F_2\to 0 \tag{3.70}$$

with $F_1\in\overline{\mathfrak{T}}_3$ and $F_2\in\overline{\mathfrak{F}}_3$. Applying $\widehat{\Psi}$, we have an exact sequence

$$\begin{aligned} 0 &\longrightarrow \widehat{\Psi}^0(F_2)\longrightarrow\widehat{\Psi}^0(\Psi^1(E))\longrightarrow\widehat{\Psi}^0(F_1)\\ &\longrightarrow \widehat{\Psi}^1(F_2)\longrightarrow\widehat{\Psi}^1(\Psi^1(E))\longrightarrow\widehat{\Psi}^1(F_1)\\ &\longrightarrow \widehat{\Psi}^2(F_2)\longrightarrow\widehat{\Psi}^2(\Psi^1(E))\longrightarrow\widehat{\Psi}^2(F_1)\longrightarrow 0.\end{aligned} \tag{3.71}$$

By Lemma 3.6.15, we have $\widehat{\Psi}^0(F_1)=\widehat{\Psi}^2(F_2)=0$.

(1) Assume that $\deg_{\min,G_1}(E)\geq 0$. By Lemma 3.6.16 (2) and Lemma 3.6.9, (a) and the first claim of (c) hold. For the second claim of (c), by Lemma 3.6.16 (2), it is sufficient to prove that $\widehat{\Psi}^2(\Psi^2(E))$ does not contain a non-trivial 0-dimensional subobject. By the exact sequence

$$0\to\widehat{\Psi}^0(\Psi^1(E))\to\widehat{\Psi}^2(\Psi^2(E))\to E \tag{3.72}$$

and the torsion-freeness of $\widehat{\Psi}^0(\Psi^1(E))$, we get our claim.

We prove (b). By Lemma 6.3.2 and (a), we have $\widehat{\Psi}^2(\Psi^1(E))=0$. Then WIT_1 holds for F_1. We have a surjective homomorphism

$$E\to\widehat{\Psi}^1(\Psi^1(E)). \tag{3.73}$$

Hence E has a quotient sheaf $\widehat{\Psi}^1(F_1)$ with $\deg_{G_1}(\widehat{\Psi}^1(F_1))=-\deg_{G_3}(F_1)\leq 0$. If $\deg_{G_1}(\widehat{\Psi}^1(F_1))<0$, then we see that $\operatorname{rk}\widehat{\Psi}^1(F_1)>0$ and $E\not\in\overline{\mathfrak{T}}_1^{\mu}$. Hence $\deg_{G_1}(\widehat{\Psi}^1(F_1))=-\deg_{G_3}(F_1)=0$. Then $F_1\in\overline{\mathfrak{T}}_3$ implies that $\operatorname{rk}\widehat{\Psi}^1(F_1)=-\chi(G_3,F_1)\leq 0$. Since $\chi(G_1,\widehat{\Psi}^1(F_1))=-\operatorname{rk}F_1\leq 0$, the G_1-twisted Hilbert polynomial of $\widehat{\Psi}^1(F_1)$ is 0. Therefore $F_1=0$.

(2) Assume that $\deg_{\max,G_1}(E)<0$. By Lemma 3.6.4 and Lemma 3.6.16, (a) and (c) hold. We prove (b). Since $\Psi^2(E)=0$, Lemma 6.3.2 implies that $\widehat{\Psi}^0\Psi^1(E)=0$. Hence WIT_1 holds for F_2 and we have

an injective morphism $\widehat{\Psi}^1(F_2) \to \widehat{\Psi}^1(\Psi^1(E)) \to E$. Since $\deg_{G_1}(\widehat{\Psi}^1(F_2)) \geq 0$, we have $\widehat{\Psi}^1(F_2) = 0$, which implies that $F_2 = 0$. □

Proof of Proposition 3.6.14.
For $E \in \overline{\mathfrak{A}}_1^\mu$, we have an exact sequence in $\overline{\mathfrak{A}}_1^\mu$

$$0 \to H^{-1}(E)[1] \to E \to H^0(E) \to 0. \tag{3.74}$$

Then we have an exact triangle

$$\Psi(H^0(E))[2] \to \Psi(E[-2]) \to \Psi(H^{-1}(E))[1] \to \Psi(H^0(E))[3]. \tag{3.75}$$

Hence $\Psi^i(E[-2]) = 0$ for $i \neq -1, 0$ and we have an exact sequence

$$\begin{aligned} 0 \longrightarrow \Psi^1(H^0(E)) \longrightarrow \Psi^{-1}(E[-2]) \longrightarrow \Psi^0(H^{-1}(E)) \\ \longrightarrow \Psi^2(H^0(E)) \longrightarrow \Psi^0(E[-2]) \longrightarrow \Psi^1(H^{-1}(E)) \longrightarrow 0. \end{aligned} \tag{3.76}$$

By Lemme 3.6.17, $\Psi^{-1}(E[-2]) \in \overline{\mathfrak{F}}_3$ and $\Psi^0(E[-2]) \in \overline{\mathfrak{T}}_3$. Therefore $\Psi(E[-2]) \in (\overline{\mathfrak{A}}_3)_{op}$. □

Definition 3.6.18. (1) Let $\mathrm{Per}(X'/Y')^D_{w_0^\vee}$ be the full subcategory of $\mathrm{Per}(X'/Y')^D$ consisting of G_3-twisted semi-stable objects E with $\deg_{G_3}(E) = \chi(G_3, E) = 0$.
(2) Let $\mathcal{C}_0$ (resp. $\mathrm{Per}(X'/Y')^D_0$) be the full subcategory of $\mathcal{C}$ (resp. $\mathrm{Per}(X'/Y')^D$) consisting of 0-dimensional objects.

Proposition 3.6.19. Ψ *induces the following correspondences:*

$$\begin{aligned} \mathcal{C}_0 &\cong (\mathrm{Per}(X'/Y')^D_{w_0^\vee})_{op}, \\ \mathcal{C}_{v_0} &\cong (\mathrm{Per}(X'/Y')^D_0)_{op}. \end{aligned} \tag{3.77}$$

Proof. By Lemma 3.6.11, $\Psi^2(\mathcal{C}_0)$ is contained in $(\mathrm{Per}(X'/Y')^D_{w_0^\vee})_{op}$. It is easy to see that $\mathrm{Per}(X'/Y')^D_{w_0^\vee}$ is generated by $\Psi^2(A_{ij}), i, j \geq 0$ and $\Psi^2(\mathbb{C}_x)$, $x \in X \setminus \cup_i Z_i$. Thus the first claim holds.
We have an equivalence

$$\begin{array}{ccc} \mathrm{Per}(X'/Y')_0 & \to & (\mathrm{Per}(X'/Y')^D_0)_{op} \\ E & \mapsto & \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(E, \mathcal{O}_X)[2]. \end{array} \tag{3.78}$$

Then the second claim is a consequence of Proposition 3.2.13 (1). □

3.7. Preservation of Gieseker stability conditions.

Proposition 3.7.1. *Let E be a G_1-twisted semi-stable object with $\deg_{G_1}(E) = 0$ and $\chi(G_1, E) < 0$. Then* WIT_1 *holds for E and $\Psi^1(E)$ is G_3-twisted semi-stable. In particular, we have an isomorphism*

$$\mathcal{M}_H^{G_1}(v)^{ss} \to \mathcal{M}_{\widehat{H}}^{G_3}(-\Psi(v))^{ss} \tag{3.79}$$

which preserves the S-equivalence classes, where $v = lv_0 + a\varrho_X + (D + (D/r_0, \xi_0)\varrho_X)$, $l > 0$, $a < 0$.

Proof. We note that $E \in \overline{\mathfrak{F}}_1 \cap \overline{\mathfrak{T}}_1^\mu$. By Lemma 3.6.4 and Lemma 3.6.17, WIT_1 holds for E and $\Psi^1(E) \in \overline{\mathfrak{F}}_3$. Assume that $\Psi^1(E)$ is not G_3-twisted stable. Then there is an exact sequence in $\mathrm{Per}(X'/Y')^D$

$$0 \to F_1 \to \Psi^1(E) \to F_2 \to 0 \tag{3.80}$$

such that F_1 is a G_3-twisted stable object with $\deg_{G_3}(F_1) = 0$ and

$$0 > \frac{\chi(G_3, F_1)}{\mathrm{rk}\, F_1} \geq \frac{\chi(G_3, \Psi^1(E))}{\mathrm{rk}\, \Psi^1(E)}, \tag{3.81}$$

and $F_2 \in \overline{\mathfrak{F}}_3$. Then we have an exact sequence

$$0 \to \widehat{\Psi}^1(F_2) \to E \to \widehat{\Psi}^1(F_1) \to 0. \tag{3.82}$$

Since

$$\begin{aligned} \frac{\chi(G_1, \widehat{\Psi}^1(F_1))}{\mathrm{rk}(\widehat{\Psi}^1(F_1))} &= \frac{\mathrm{rk}\, F_1}{\chi(G_3, F_1)} \\ &\leq \frac{\mathrm{rk}\, \Psi^1(E)}{\chi(G_3, \Psi^1(E))} = \frac{\chi(G_1, E)}{\mathrm{rk}\, E}, \end{aligned} \tag{3.83}$$

we have

$$\frac{\chi(G_3, F_1)}{\mathrm{rk}\, F_1} = \frac{\chi(G_3, \Psi^1(E))}{\mathrm{rk}\, \Psi^1(E)}. \tag{3.84}$$

Hence $\Psi^1(E)$ is G_3-twisted semi-stable. Thus we have a morphism $\mathcal{M}_H^{G_1}(v)^{ss} \to \mathcal{M}_{\widehat{H}}^{G_3}(-\Psi(v))^{ss}$. It is easy to see that this morphism preserves the S-equivalence classes. By the symmetry of the conditions, we have the inverse morphism, which shows the second claim. □

The following is a generalization of [Y5, Thm. 1.7].

Proposition 3.7.2. *Let $w \in v(\mathbf{D}(X'))$ be a Mukai vector such that $\langle w^2 \rangle \geq -2$ and*

$$w = lw_0 + a\varrho_{X'} + \left(d\widehat{H} + \widehat{D} + \frac{1}{r_0}(d\widehat{H} + \widehat{D}, \xi_0)\varrho_{X'}\right), \tag{3.85}$$

where $l \geq 0$, $a > 0$ and $D \in \mathrm{NS}(X) \otimes \mathbb{Q} \cap H^{\perp}$. Assume that

$$\begin{aligned} d &> \max\{(4l^2r_0^3 + 1/(H^2)), 2r_0^2 l(\langle w^2 \rangle - (D^2))\}, \text{ if } l > 0, \\ a &> \max\{(2r_0 + 1), (\langle w^2 \rangle - (D^2))/2 + 1\}, \text{ if } l = 0. \end{aligned} \tag{3.86}$$

Then

1. $\mathcal{M}_H^{G_1}(\widehat{\Phi}(w))^{ss} \cong \mathcal{M}_{\widehat{H}}^{G_2}(w)^{ss}$.
2. $\mathcal{M}_H^{G_1}(\widehat{\Phi}(w))^{ss}$ *consists of local projective generators.*
3. *If $(\widehat{H}, G_2)$ is general with respect to w, then $\mathcal{M}_H^{G_1}(\widehat{\Phi}(w))^{ss} \cong \mathcal{M}_{H+\epsilon}^{G_1}(\widehat{\Phi}(w))^{ss}$ for a sufficiently small relatively ample divisor ϵ.*

Proof. (1) We first note that $\mathcal{F}_{\mathcal{E}}$ in [Y5] corresponds to $\widehat{\Phi}$. Since [Y5, Thm. 2.1, Thm. 2.2] are replaced by Theorem 3.5.8, 3.6.1 and since [Y5, Prop. 2.8, Prop. 2.11] also hold for our case, the same proof of [Y5, Thm. 1.7] works for our case. More precisely, in order to show that $\Phi(F), F \in \mathcal{M}_H^{G_1}(\widehat{\Phi}(w))$ does not contain a 0-dimensional subobject, we use the fact that WIT_0 holds for 0-dimensional object $E \in \mathrm{Per}(X'/Y')$ (see Proposition 3.2.13 (1)).

(2) The proof is the same as in the proof of [Y5, Rem. 2.3]. Let E be a μ-semi-stable object of $\mathcal{C}$ such that $v(E) = \widehat{\Phi}(w)$. If $\mathrm{Ext}^1(S, E) \neq 0$ for an irreducible object S of $\mathcal{C}$, then a non-trivial extension

$$0 \to E \to E' \to S \to 0 \tag{3.87}$$

gives a μ-semi-stable object E' with $\chi(G_1, E') > \chi(G_1, E)$. By Proposition [Y5, Prop. 2.8, Prop. 2.11], we get a contradiction. Hence $\mathrm{Ext}^1(E, S) \cong \mathrm{Ext}^1(S, E)^{\vee} = 0$ for any irreducible object S of $\mathcal{C}$. Since $\mathrm{Ext}^2(E, S) \cong \mathrm{Hom}(S, E)^{\vee} = 0$, it is sufficient to prove that $\chi(S, E) > 0$. We note that $\chi(S, E) = \chi(S, \widehat{\Phi}(w)) = a\chi(S, G_1) + (c_1(S), D)$. Since $(H, c_1(S)) = 0$, we have $|(c_1(S), D)^2| \leq |(c_1(S)^2)(D^2)| = -2(D^2)$. Since $\chi(S, G_1) > 0$, it is sufficient to prove that $a > \sqrt{-2(D^2)}$.

We first assume that $l > 0$. Then $d(H^2) - 1 > 4l^2r_0^3(H^2)$ and $d > 2r_0^2 l(\langle w^2 \rangle - (D^2)) = 2r_0^2 l(d^2(H^2) - 2lar_0)$. Hence

$$a > \frac{d(d(H^2) - 1/(2r_0^2 l))}{2r_0 l} > \frac{d}{2lr_0} 4l^2r_0^3(H^2) = 2dlr_0^2(H^2). \tag{3.88}$$

Hence $a > 2(4l^2r_0^3)lr_0^2(H^2) = 8r_0(lr_0)^3 r_0(H^2) \geq 8$. If $-(D^2) \leq 4$, then $a > 3 > \sqrt{-2(D^2)}$. If $-(D^2) > 4$, then $\langle w^2 \rangle - (D^2) \geq -2 - (D^2) > -(D^2)/2$. Hence

$$a > 2dlr_0^2(H^2) > r_0(\langle w^2 \rangle - (D^2))4(lr_0)^2 r_0(H^2) > \sqrt{-2(D^2)}. \tag{3.89}$$

We next assume that $l = 0$. Then $a > 2r_0 + 1$ and $a > \langle w^2 \rangle/2 + 1 - (D^2)/2 \geq -(D^2)/2$. If $-(D^2) \geq 8$, then $a > -(D^2)/2 \geq \sqrt{-2(D^2)}$. If $-(D^2) < 8$, then since $a \geq 2r_0 + 1 + 1/r_0$, $\sqrt{-2(D^2)} < 4 \leq a$.

Therefore $\chi(E, S) > 0$ and E is a local projective generator of $\mathcal{C}$.

(3) By our assumption, $\mathcal{M}_H^{G_1}(\widehat{\Phi}(w))^{ss} = \mathcal{M}_H^{G_1}(\widehat{\Phi}(w))^{\mu\text{-}ss}$ ([Y5, Cor. 2.14]) and H is a general polarization. Hence for $E \in \mathcal{M}_H^{G_1}(\widehat{\Phi}(w))^{ss}$ and a subobject E_1 of E, $\frac{(c_1(E),H)}{\mathrm{rk}\, E} = \frac{(c_1(E_1),H)}{\mathrm{rk}\, E_1}$ implies $\frac{c_1(E)}{\mathrm{rk}\, E} = \frac{c_1(E_1)}{\mathrm{rk}\, E_1}$. Let E be a μ-semi-stable sheaf of $v(E) = \widehat{\Phi}(w)$ with respect to H. We shall prove that $E \in \mathcal{C}$. We set

$$\Sigma := \{A_{ij}[-1] | i, j\} \cap \mathrm{Coh}(X)$$

as in Proposition 1.1.19. We assume that $\mathrm{Hom}(E, F) \neq 0$ for $F \in \Sigma$. Then there is a μ-semi-stable sheaf $E' \in \mathcal{C} \cap \mathrm{Coh}(X)$ with respect to H fitting in an exact sequence

$$0 \to E' \to E \to F' \to 0, \tag{3.90}$$

where $F' \in \mathcal{C}[-1] \cap \mathrm{Coh}(X)$. Then we see that $\chi(G_1, E') > \chi(G, E)$, which is a contradiction. Therefore $E \in \mathcal{C}$. Then we can easily see that E is μ-semi-stable in $\mathcal{C}$. □

Corollary 3.7.3. *If (G, H) is general with respect to v, then $M_H^G(v)$ is isomorphic to the moduli space of usual stable sheaves on a $K3$ surface.*

Proof. We first construct a primitive and isotropic Mukai vector u such that $\operatorname{rk} u > 0$ and $(\operatorname{rk} G)c_1(u) - (\operatorname{rk} u)c_1(G^\vee) \in \mathbb{Z}H$: We first take a primitive isotropic Mukai vector t such that $t = lv(G^\vee) + a\varrho_X$. Then for a sufficiently small τ, $T := M_H^{G^\vee+\tau}(t)$ is a $K3$ surface. Let $\mathcal{F}$ be the universal family on $T \times X$ as a twisted object. Then we have an equivalence $\Phi_{X\to T}^{\mathcal{F}^\vee} : \mathbf{D}(X) \to \mathbf{D}^\beta(T)$. We consider $\Pi := \Phi_{T\to X}^{\mathcal{F}(nD)} \circ \Phi_{X\to T}^{\mathcal{F}^\vee} : \mathbf{D}(X) \to \mathbf{D}(X)$, $n \gg 0$, where we set $D := \widehat{H}$. Then Π also induces a Hodge isometry $\Pi : H^*(X,\mathbb{Z}) \to H^*(X,\mathbb{Z})$. By its construction, Π preserves the subspace $(\mathbb{Q}t + \mathbb{Q}H + \mathbb{Q}\varrho_X) \cap H^*(X,\mathbb{Z})$ and $\operatorname{rk}\Pi(\varrho_X) > 0$ for $n \gg 0$. Hence $u := \Pi(\varrho_X)$ satisfies the claim. Since $c_1(u)/\operatorname{rk} u - c_1(G^\vee)/\operatorname{rk} G^\vee \in \mathbb{Q}H$, $\chi(u, A_{ij}^\vee[2])/\operatorname{rk} u = \chi(G^\vee, A_{ij}^\vee[2])/\operatorname{rk} G$. By Corollary 2.4.4, there is a local projective generator G_u of $\mathcal{C}^D$ with $v(G_u) = 2u$. Since $\langle \Pi(\mathcal{O}_X), u\rangle = -1$, $X_1 := M_H^{u+\alpha}(u)$ is a fine moduli space of stable objects of $\mathcal{C}^D$. Since $\mathcal{C}$ satisfies Assumption 3.1.1, $\mathcal{C}^D$ also satisfies Assumption 3.1.1. Let $\mathcal{E}$ be the universal family on $X \times X_1$. By Theorem 3.6.1, we can regard $\mathcal{E}$ as a universal family of $v_0 + \gamma$-twisted stable objects of $\operatorname{Per}(X_1/Y_1)^D$ with respect to H_1, where $Y_1 := \overline{M}_H^u(u)$, $H_1 := \widehat{H}$, $v_0 = v(\mathcal{E}_{|\{x\}\times X_1})$ and γ is determined by α. Then $(M_{H_1}^{v_0+\gamma}(v_0), \widehat{H}_1) = (X,H)$. For $\widehat{\Phi} = \Phi_{X\to X_1}^{\mathcal{E}}$ and $\mathcal{M}_H^{u^\vee}(ve^{mH})^{ss}$, $m \gg 0$, we shall apply Proposition 3.7.2. Then $\mathcal{M}_H^{u^\vee}(v)^{ss}$ is isomorphic to a moduli stack of usual semi-stable sheaves on X_1. Since $\mathcal{M}_H^{u^\vee}(v)^{ss} = \mathcal{M}_H^G(v)^{ss}$, we get our claim. □

Since (3.86) is numerical, we can apply Proposition 3.7.2 to a family of $K3$ surfaces.

Example 3.7.4. Let $f : (\mathcal{X}, \mathcal{H}) \to S$ be a family of polarized $K3$ surfaces over S. Let $v_0 := (r, d\mathcal{H}, a)$, $\gcd(r,a) = 1$ be a family of isotropic Mukai vectors. We set $\mathcal{X}' := M_{\mathcal{X}/S}^{v_0}(v_0)$. Then we have a family of polarizations $\mathcal{H}'$ on $\mathcal{X}'$. Since $\gcd(r,a) = 1$, there is a universal family $\mathcal{E}$ on $\mathcal{X}' \times_S \mathcal{X}$ and we have a family of Fourier-Mukai transforms $\Phi_{\mathcal{X}\to\mathcal{X}'}^{\mathcal{E}} : \mathbf{D}(\mathcal{X}) \to \mathbf{D}(\mathcal{X}')$. Then we can apply Proposition 3.7.1 and Proposition 3.7.2 to families of moduli spaces over S.

We also give a generalization of [Y1, Thm. 7.6] based on Theorem 3.5.8 and Proposition 3.6.14. We set

$$d_{\min} := \min\{\deg_{G_1}(F) > 0 | F \in \mathbf{D}(X)\}. \tag{3.91}$$

Proposition 3.7.5. *Assume that $\mathfrak{T}_1 = \mathfrak{T}_1^\mu$. Let $v \in H^*(X,\mathbb{Z})$ be a Mukai vector of a complex such that $\deg_{G_1}(v) = d_{\min}$.*

(1) *If $\operatorname{rk}\Phi(v) \le 0$, then Φ induces an isomorphism*

$$\mathcal{M}_H^{G_1}(v)^{ss} \to \mathcal{M}_{\widehat{H}}^{G_2}(-\Phi(v))^{ss} \tag{3.92}$$

by sending E to $\Phi^1(E)$.

(2) *If $\operatorname{rk}\Psi(v) \ge 0$, then Ψ induces an isomorphism*

$$\mathcal{M}_H^{G_1}(v)^{ss} \to \mathcal{M}_{\widehat{H}}^{G_3}(\Psi(v))^{ss} \tag{3.93}$$

by sending E to $\Psi^2(E)$.

The proof is an easy exercise. We shall give a proof in [MYY], as an application of Bridgeland's stability condition.

Remark 3.7.6. In [Y6], we constructed actions of Lie algebras on the cohomology groups of some moduli spaces of stable sheaves. In particular, we constructed the action on the cohomology groups of some moduli spaces of stable objects of ${}^{-1}\operatorname{Per}(X/Y)$ in [Y6, Prop. 6.15]. Then a generalization of [Y6, Prop. 6.15] to the objects in $\operatorname{Per}(X'/Y')$ corresponds to the action in [Y6, Example 3.1.1] via Proposition 3.7.5.

4. Fourier-Mukai transforms on elliptic surfaces.

4.1. **Moduli of stable sheaves of dimension 2.** Let $Y \to C$ be a morphism from a normal projective surface to a smooth curve C such that a general fiber is an elliptic curve. Let $\pi : X \to Y$ be the minimal resolution. Then $\mathfrak{p} : X \to C$ is an elliptic surface over a curve C. We fix a divisor H on X which is the pull-back of an ample divisor on Y. As in section 3, let $\mathcal{C}$ be the category in Lemma 1.1.5 satisfying Assumption 3.1.1. We also use the notation A_{ij} in section 3. Let G_1 be a locally free sheaf on X which is a local projective generator of $\mathcal{C}$. Let $\mathbf{e} \in K(X)_{\text{top}}$ be the topological invariant of a locally free sheaf E of rank r and degree d on a fiber of $\mathfrak{p}$. Thus $\operatorname{ch}(\mathbf{e}) = (0, rf, d)$, where f is a fiber of $\mathfrak{p}$. Assume that $\mathbf{e}$ is primitive. Then $\overline{M}_H^{G_1}(\mathbf{e})$ consists of G_1-twisted stable objects, if $G_1 \in K(X)_{\text{top}} \otimes \mathbb{Q}$, $\operatorname{rk} G_1 > 0$ is general with respect to $\mathbf{e}$ and H. From now on, we assume that $\chi(G_1, \mathbf{e}) = 0$. By [O-Y, sect. 1.1], we do not lose generality.

Remark 4.1.1. We have $\overline{M}_H^{G_1}(\mathbf{e}) = \overline{M}_{H+nf}^{G_1}(\mathbf{e})$ for all n.

Lemma 4.1.2. *We set*

$$\mathbf{e}^\perp := \{E \in K(X)_{\text{top}} | \chi(E, \mathbf{e}) = 0\}. \tag{4.1}$$

(1) $-\chi(\ ,\ )$ *is symmetric on* $\mathbf{e}^\perp$.
(2) $M := (\mathbb{Z}\tau(G_1)+\mathbb{Z}\tau(\mathbb{C}_x)+\mathbb{Z}\mathbf{e})^\perp/\mathbb{Z}\mathbf{e}$ *is a negative definite even lattice of rank* $\rho(X)-2$.

Proof. (1) For a divisor D, we set

$$\nu(D) := \tau(\mathcal{O}_X(D)-\mathcal{O}_X) - \frac{\chi(G_1, \mathcal{O}_X(D)-\mathcal{O}_X)}{\operatorname{rk} G_1}\tau(\mathbb{C}_x) \in K(X)_{\mathrm{top}}\otimes\mathbb{Q}. \tag{4.2}$$

Then ν induces a homomorphism

$$\mathrm{NS}(X)\otimes\mathbb{Q} \to K(X)_{\mathrm{top}}\otimes\mathbb{Q} \tag{4.3}$$

such that $\operatorname{rk}(\nu(D))=0$, $c_1(\nu(D))=D$ and $\chi(G_1,\nu(D))=0$. For $E\in K(X)\otimes\mathbb{Q}$, we have an expression

$$\tau(E) = l\tau(G_1)+a\tau(\mathbb{C}_x)+\nu(D) \tag{4.4}$$

where $l,a\in\mathbb{Q}$ and $D\in\mathrm{NS}(X)\otimes\mathbb{Q}$. If $\chi(E,\mathbf{e})=0$, then D satisfies $(D,f)=0$. Hence we have a decomposition

$$\mathbf{e}^\perp\otimes\mathbb{Q} = (\mathbb{Q}\tau(G_1)+\mathbb{Q}\tau(\mathbb{C}_x)) + \nu((\mathbb{Q}f)^\perp). \tag{4.5}$$

For $E,F\in K(X)$, we have

$$\chi(E,F)-\chi(F,E) = (\operatorname{rk} E c_1(F) - \operatorname{rk} F c_1(E), K_X). \tag{4.6}$$

Hence the claim (1) holds.

(2) By (4.5), the signature of $\mathbf{e}^\perp/\mathbb{Z}\mathbf{e}$ is $(1,\rho(X)-1)$. We note that $\mathbb{Q}\tau(G_1)+\mathbb{Q}\tau(\mathbb{C}_x)\to(\mathbf{e}^\perp/\mathbb{Z}\mathbf{e})\otimes\mathbb{Q}$ is injective and defines a subspace of signature $(1,1)$. Hence M is negative definite. Since $(\mathbb{Z}\tau(\mathbb{C}_x)+\mathbb{Z}\mathbf{e})^\perp$ is an even lattice, we get our claim. □

Lemma 4.1.3. (1) *Assume that* G_1 *is general with respect to* $\mathbf{e}$ *and* H. *Then* $\overline{M}_H^{G_1}(\mathbf{e})$ *is a smooth elliptic surface over* C *and* $E\otimes K_X\cong E$ *for all* $E\in\overline{M}_H^{G_1}(\mathbf{e})$.
(2) *Let* E *be a* G_1*-twisted stable object such that* $\operatorname{Supp}(E)\subset\mathfrak{p}^{-1}(c)$, $c\in C$. *If* $\chi(G_1,E)=0$ *and* $(c_1(E),H)<(c_1(\mathbf{e}),H)$, *then* $\chi(E,E)=2$ *and* $E\otimes K_X\cong E$.

Proof. (1) In [Br1, Thm. 1.2], Bridgeland proved that $\overline{M}_H^{G_1}(\mathbf{e})$ is smooth and defines a Fourier-Mukai transform $\mathbf{D}(\overline{M}_H^{G_1}(\mathbf{e}))\to\mathbf{D}(X)$, if $G_1=\mathcal{O}_X$ is general with respect to $\mathbf{e}$ and H. We can easily generalize the arguments in [Br1, sect. 4] to the moduli space $\overline{M}_H^{G_1}(\mathbf{e})$ of G_1-twisted semi-stable objects, if G_1 is general with respect to $\mathbf{e}$ and H. Then the claims follow.

(2) Since $\operatorname{Supp}(E)\subset\mathfrak{p}^{-1}(c)$ and $\chi(G_1,E)=0$, we have $E\in(\mathbb{Z}\tau(\mathbb{C}_x)+\mathbb{Z}\tau(G_1)+\mathbb{Z}\mathbf{e})^\perp$. Since $(c_1(E),H)<(c_1(\mathbf{e}),H)$, we get

$$2\le\chi(E,E)=\dim\operatorname{Hom}(E,E)+\dim\operatorname{Hom}(E,E\otimes K_X)-\dim\operatorname{Ext}^1(E,E). \tag{4.7}$$

Hence $\operatorname{Hom}(E,E\otimes K_X)\neq0$. Since $K_X^{\otimes m}\in\mathfrak{p}^*(\operatorname{Pic}(C))$ for an integer m, we see that $E\otimes K_X$ is a G_1-twisted stable object with $\tau(E)=\tau(E\otimes K_X)$, which implies that $E\otimes K_X\cong E$ and $\chi(E,E)=2$. □

In the same way as in the proof of Theorem 3.1.5, we get the following results.

Corollary 4.1.4. (1) $\overline{M}_H^{G_1}(\mathbf{e})$ *is a normal surface and the singular points* $q_1,q_2,\dots,q_m$ *of* $\overline{M}_H^{G_1}(\mathbf{e})$ *correspond to the* S*-equivalence classes of properly* G_1*-twisted semi-stable objects.*
(2) *Let* $\oplus_{j=0}^{s_i'}E_{ij}^{\oplus a_{ij}'}$ *be the* S*-equivalence class corresponding to* q_i. *Then the matrix* $(\chi(E_{ij},E_{ik}))_{j,k\ge0}$ *is of affine type* $\tilde{A},\tilde{D},\tilde{E}$. *We assume that* $a_{i0}=1$ *for all* i. *Then* $q_1,q_2,\dots,q_m$ *are rational double points of type* A,D,E *according as the type of the matrices* $(\chi(E_{ij},E_{ik}))_{j,k\ge1}$.
(3) *We take a sufficiently small general* $\alpha\in K(X)\otimes\mathbb{Q}$ *such that* $\chi(\alpha,\mathbf{e})=0$. *Then* $\pi':\overline{M}_H^{G_1+\alpha}(\mathbf{e})\to\overline{M}_H^{G_1}(\mathbf{e})$ *is the minimal resolution.*
(4) *Assume that* $a_{i0}'=1$ *for all* i *and* $\chi(\alpha,E_{ij})<0$ *for all* $j>0$. *We set*

$$C_{ij}' := \{E\in M_H^{G_1+\alpha}(\mathbf{e}) \mid \operatorname{Hom}(E_{ij},E)\neq0\}. \tag{4.8}$$

Then C_{ij}' *is a smooth rational curve such that* $(C_{ij}',C_{i'j'}')=-\chi(E_{ij},E_{i'j'})$ *and* ${\pi'}^{-1}(q_i)=\sum_{j\ge1}a_{ij}'C_{ij}'$.

Remark 4.1.5. In Theorem 3.1.5, we assume that $\chi(\alpha,E_{ij})>0$. So the definition of C_{ij}' is different from that in Lemma 3.2.4. For the smoothness of C_{ij}', we use the moduli of coherent systems (E,V), where $E\in M_H^{G_1+\alpha}(\mathbf{e})$ and V is a 1-dimensional subspace of $\operatorname{Hom}(E_{ij},E)$.

From now on, we take an α in Corollary 4.1.4 (3) and set $X':=\overline{M}_H^{G_1+\alpha}(\mathbf{e})$, $Y':=\overline{M}_H^{G_1}(\mathbf{e})$. Let $\mathfrak{q}:X'\to C$ be the structure morphism of the elliptic fibration.

4.2. **Fourier-Mukai duality for an elliptic surface.** Let $\mathcal{E}$ be a universal family as a twisted sheaf on $X' \times X$. For simplicity, we assume that it is an untwisted sheaf. We set

$$\begin{aligned}\Psi(E) :=& \mathbf{R}\operatorname{Hom}_{p_{X'}}(p_X^*(E), \mathcal{E}) = \Phi(E)^\vee[-2],\ E \in \mathbf{D}(X),\\ \widehat{\Psi}(F) :=& \mathbf{R}\operatorname{Hom}_{p_X}(p_{X'}^*(F), \mathcal{E}),\ F \in \mathbf{D}(X').\end{aligned} \tag{4.9}$$

Lemma 4.2.1. *Replacing G_1 by $G_1 - n\mathbb{C}_x$, $n \gg 0$, we can choose $\det \Psi(G_1)^\vee \in \operatorname{Pic}(X')$ as the pull-back of an ample line bundle on W. Let $\widehat{H}$ be a divisor with $\mathcal{O}_{X'}(\widehat{H}) = \det \Psi(G_1)^\vee$.*

Proof. We note that $\det \Psi(\mathbb{C}_x) = rf$. Hence $\det \Psi(G_1 - n\mathbb{C}_x)^\vee = \det \Psi(G_1)^\vee(nrf)$. We set

$$\xi := mr \operatorname{rk} G_1 (H, f)(-G_1^\vee + (\operatorname{rk} G_1) n(n+m)(H^2)/2\varrho_X). \tag{4.10}$$

By (1.104), $\det p_{X'!}(\mathcal{E} \otimes p_X^*(\xi))$ is the pull-back of a polarization of Y' for $m \gg n \gg 0$. Since $\det \Psi(\xi^\vee) = \det p_{X'!}(\mathcal{E} \otimes p_X^*(\xi))$ and $-\operatorname{ch}(\xi^\vee) \equiv mr \operatorname{rk} G_1 (H, f) \operatorname{ch}(G_1) \mod \mathbb{Q}\varrho_X$, we get our claim. □

Lemma 4.2.2. *We set $A'_{ij} := \Psi(E_{ij})[2]$.*

(1) *There are $\mathbf{b}'_i := (b'_{i1}, b'_{i2}, \dots, b'_{is'_i})$, $i = 1, \dots, m$ such that*

$$\begin{aligned}A'_{ij} &= \mathcal{O}_{C'_{ij}}(b'_{ij})[1],\ j > 0\\ A'_{i0} &= A_0(\mathbf{b}'_i).\end{aligned} \tag{4.11}$$

(2) *Irreducible objects of $\operatorname{Per}(X'/Y', \mathbf{b}'_1, ..., \mathbf{b}'_m)$ are*

$$A'_{ij} (1 \le i \le m, 0 \le j \le s'_i),\ \mathbb{C}_{x'} (x' \in X' \setminus \cup_i Z'_i). \tag{4.12}$$

Proof. It is sufficient to prove (1) by Proposition 1.2.19. By the choice of α, we have

$$\begin{aligned}\operatorname{Ext}^2(E_{ij}, \mathcal{E}_{|\{x'\}\times X}) =& 0,\ j > 0,\\ \operatorname{Hom}(E_{i0}, \mathcal{E}_{|\{x'\}\times X}) =& 0\end{aligned} \tag{4.13}$$

for all $x' \in X'$. Then the claim for $j > 0$ follow from the proof of Corollary 4.1.4 (4). For $x' \in {\pi'}^{-1}(q_i)$, we have an exact sequence

$$0 \to F_i \to \mathcal{E}_{|\{x'\}\times X} \to E_{i0} \to 0, \tag{4.14}$$

where F_i is a G_1-twisted semi-stable object which is S-equivalent to $\oplus_{j>0} E_{ij}^{\oplus_j a'_{ij}}$. Applying Ψ, we have an exact sequence

$$0 \to \Psi(F_i)[1] \to A'_{i0} \to \mathbb{C}_{x'} \to 0. \tag{4.15}$$

It is easy to see that

$$\operatorname{Hom}(A'_{i0}, A'_{ij}[-1]) = \operatorname{Ext}^1(A'_{i0}, A'_{ij}[-1]) = 0. \tag{4.16}$$

By Lemma 2.1.8, we get $A'_{i0} = A_0(\mathbf{b}'_i)$. □

We define $\operatorname{Per}(X'/Y')$ and $\operatorname{Per}(X'/Y')^D$ as in subsection 3.2. Replacing G_1 by G'_1 with $\tau(G'_1) = \tau(G_1) - n\tau(\mathbb{C}_x)$, we may assume that $G_{1|\mathfrak{p}^{-1}(t)}$, $t \in C$ is a stable vector bundle for a general $t \in C$. Then $L'_2 = \Psi(G_1)[1]$ is a torsion object of $\operatorname{Per}(X'/Y') \cap \operatorname{Coh}(X')$ such that $c_1(L_2) = \widehat{H}$. Indeed L'_2 is a coherent torsion sheaf on X'. Since $\operatorname{Hom}(L'_2, A'_{ij}[-1]) = \operatorname{Hom}(E_{ij}, G_1) = 0$, $L'_2 \in \operatorname{Per}(X'/Y')$.

Lemma 4.2.3. *Let L_1 be a line bundle on a smooth curve $C \in |H|$ and set $G_2 := \Psi(L_1)[1]$. Then we have*

$$\begin{aligned}\operatorname{Hom}(G_2, \mathbb{C}_{x'}[k]) &= 0, \quad k \ne 0,\\ \operatorname{Hom}(G_2, A'_{ij}[k]) &= 0, \quad k \ne 0,\\ \dim \operatorname{Hom}(G_2, A'_{ij}) &= (c_1(E_{ij}), H).\end{aligned} \tag{4.17}$$

In particular G_2 is a local projective generator of $\operatorname{Per}(X'/Y')$.

Proof. The claim follows from the following relations:

$$\begin{aligned}\operatorname{Hom}(G_2, \mathbb{C}_{x'}[k]) &= \operatorname{Hom}(\Psi(L_1)[1], \Psi(\mathcal{E}_{|\{x'\}\times X})[2+k])\\ &= \operatorname{Hom}(\mathcal{E}_{|\{x'\}\times X}, L_1[k+1]),\\ \operatorname{Hom}(G_2, A'_{ij}[k]) &= \operatorname{Hom}(\Psi(L_1)[1], \Psi(E_{ij})[2+k])\\ &= \operatorname{Hom}(E_{ij}, L_1[k+1]).\end{aligned} \tag{4.18}$$

□

For a conveniense sake, we summalize the image of $\mathbb{C}_x[-2], \mathcal{E}_{|\{x'\}\times X}, G_1, L_1$ by Ψ:

$$
\begin{aligned}
\Psi(\mathbb{C}_x[-2]) &= \mathcal{E}_{|X'\times\{x\}}, \\
\Psi(\mathcal{E}_{|\{x'\}\times X}) &= \mathbb{C}_{x'}[-2], \\
\Psi(G_1) &= L_2[-1], \\
\Psi(L_1) &= G_2[-1].
\end{aligned}
\tag{4.19}
$$

Definition 4.2.4. We set $\Psi^i(E) := {}^pH^i(\Psi(E)) \in \operatorname{Per}(X'/Y')$ and $\widehat{\Psi}^i(E) := {}^pH^i(\widehat{\Psi}(E)) \in \operatorname{Per}(X/Y)$.

Lemma 4.2.5. WIT_2 *with respect to* Ψ *holds for all 0-dimensional objects* E *of* $\operatorname{Per}(X'/Y')$ *and* $\Psi^2(E)$ *is* G_2*-twisted semi-stable. Moreover if* E *is an irreducible object, then* $\Psi(E)[2]$ *is a* G_2*-twisted stable object of* $\operatorname{Per}(X'/Y')$.

Proof. It is sufficient to prove the claim for all irreducible objects E of $\mathcal{C}$. Since $\mathcal{E}_{|\{x'\}\times X}$ and E_{ij} are purely 1-dimensional objects of $\mathcal{C}$, $\operatorname{Hom}(E, \mathcal{E}_{|\{x'\}\times X}) = \operatorname{Hom}(E, E_{ij}) = 0$ for all $x' \in X'$ and i, j. Hence $\Psi^1(E)$ is a torsion free object of $\mathfrak{C}_2$. Since $\operatorname{Hom}(E, \mathcal{E}_{|\{x'\}\times X}[1]) = 0$ if $\operatorname{Supp}(E) \cap \mathfrak{p}^{-1}(\mathfrak{p}(x')) = \emptyset$, $\Psi^1(E) = 0$. Therefore WIT_2 holds for all 0-dimensional objects of $\operatorname{Per}(X'/Y')$.

For the G_2-twisted stability of $\Psi(E)[2]$, we first note that $\chi(G_2, \Psi(E)[2]) = \chi(\Psi(L_1)[1], \Psi(E)[2]) = \chi(E, L_1[1]) = 0$. Assume that there is an exact sequence

$$
0 \to F_1 \to \Psi^2(E) \to F_2 \to 0 \tag{4.20}
$$

such that $0 \neq F_1 \in \operatorname{Per}(X'/Y')$ and $F_2 \in \operatorname{Per}(X'/Y')$ with $\chi(G_2, F_2) \leq 0$. Applying $\widehat{\Psi}$ to this exact sequence, we get a long exact sequence

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \widehat{\Psi}^0(F_2) & \longrightarrow & 0 & \longrightarrow & \widehat{\Psi}^0(F_1) \\
 & \longrightarrow & \widehat{\Psi}^1(F_2) & \longrightarrow & 0 & \longrightarrow & \widehat{\Psi}^1(F_1) \\
 & \longrightarrow & \widehat{\Psi}^2(F_2) & \longrightarrow & E & \longrightarrow & \widehat{\Psi}^2(F_1) \longrightarrow 0.
\end{array}
\tag{4.21}
$$

Since $\widehat{\Psi}^0(F_1) = 0$, WIT_2 holds for F_2. Since $0 \geq \chi(G_2, F_2) = \chi(\widehat{\Psi}(F_2), \widehat{\Psi}(G_2)) = \chi(\widehat{\Psi}(F_2), L_1[-1]) = (H, c_1(\widehat{\Psi}^2(F_2))) \geq 0$, we get $\chi(G_2, F_2) = 0$ and $\widehat{\Psi}^2(F_2)$ is a 0-dimensional object. Then $\widehat{\Psi}^1(F_1)$ is also 0-dimensional. Since E is an irreducible object of $\mathfrak{C}_1$, we have (i) $\widehat{\Psi}^2(F_1) = 0$ or (ii) $\widehat{\Psi}^2(F_1) \cong E$. Since WIT_2 holds for $\widehat{\Psi}^1(F_1)$ with respect to Ψ, the first case does not hold. If $\widehat{\Psi}^2(F_1) \cong E$, then $\widehat{\Psi}^1(F_1) \cong \widehat{\Psi}^2(F_2)$. Since $\widehat{\Psi}^0(F_1) = 0$, Lemma 6.3.2 implies that $\Psi^2(\widehat{\Psi}^1(F_1)) = 0$, which implies that $F_2 = \Psi^2(\widehat{\Psi}^2(F_2)) = 0$. Therefore $\Psi^2(E)$ is G_2-twisted stable. □

Theorem 4.2.6. *We set* $\mathbf{f} := \tau(\mathcal{E}_{|X'\times\{x\}})$. *Then* $\mathcal{E}_{|X'\times\{x\}}$ *is* $G_2 - \Psi(\beta)$*-twisted stable for all* $x \in X$ *and we have an isomorphism* $X \to M_{\widehat{H}}^{G_2-\Psi(\beta)}(\mathbf{f})$ *by sending* $x \in X$ *to* $\mathcal{E}_{|X'\times\{x\}} \in M_{\widehat{H}}^{G_2-\Psi(\beta)}(\mathbf{f})$.

Proof. By Lemma 4.2.5, $\mathcal{E}_{|X'\times\{x\}}$ is G_2-twisted semi-stable. If $\mathcal{E}_{|X'\times\{x\}}$ is not G_2-twisted stable, then $\mathcal{E}_{|X'\times\{x\}}$ is S-equivalent to $\oplus_j \Psi^2(A_{ij})^{\oplus a_{ij}}$. Let $F_1 \neq 0$ be a G_2-twisted stable subobject of $\mathcal{E}_{|X'\times\{x\}}$ such that $\chi(G_2, F_1) = 0$. Then F_1 is S-equivalent to $\oplus_j \Psi^2(A_{ij})^{\oplus b_{ij}}$ and $\widehat{\Psi}(F_1)[2]$ is a quotient object of $\mathbb{C}_x$. Since $\mathbb{C}_x$ is β-stable, $0 < \chi(\beta, \widehat{\Psi}(F_1)) = \chi(\Psi(\beta), F_1)$. Therefore $\mathcal{E}_{|X'\times\{x\}}$ is $G_2 - \Psi(\beta)$-twisted stable. Then we have an injective morphism $\phi : X \to \overline{M}_{\widehat{H}}^{G_2-\Psi(\beta)}(\mathbf{f})$ by sending $x \in X$ to $\mathcal{E}_{|X'\times\{x\}}$. By a standard argument, we see that ϕ is an isomorphism. □

4.3. **Tiltings of** $\mathcal{C}$**,** $\operatorname{Per}(X'/Y')$ **and their equivalence.** We set $\mathfrak{C}_1 := \mathcal{C}$ and $\mathfrak{C}_2 := \operatorname{Per}(X'/Y')$. In this subsection, we define tiltings $\overline{\mathfrak{A}}_1$, $\widehat{\mathfrak{A}}_2$ of $\mathfrak{C}_1$, $\mathfrak{C}_2$ and show that Ψ induces a (contravariant) equivalence between them. We first define the relative twisted degree of $E \in \mathfrak{C}_i$ by $\deg_{G_i}(E) := (c_1(G_i^\vee \otimes E), f)$, and define $\mu_{\max,G_i}(E)$, $\mu_{\min,G_i}(E)$ in a similar way.

Definition 4.3.1.
1. Let $\overline{\mathfrak{T}}_i$ be the full subcategory of $\mathfrak{C}_i$ consisting of objects E such that (i) E is a torsion object or (ii) E is torsion free and $\mu_{\min,G_i}(E) \geq 0$.
2. Let $\overline{\mathfrak{F}}_i$ be the full subcategory of $\mathfrak{C}_i$ consisting of objects E such that (i) $E = 0$ or (ii) E is torsion free and $\mu_{\max,G_i}(E) < 0$.

Definition 4.3.2.
1. Let $\widehat{\mathfrak{T}}_i$ be the full subcategory of $\mathfrak{C}_i$ consisting of objects E such that $\operatorname{Supp}(E)$ is contained in fibers and there is no quotient object $E \to E'$ with $\chi(G_i, E') < 0$.
2. We set

$$
\begin{aligned}
\widehat{\mathfrak{F}}_i :=& (\widehat{\mathfrak{T}}_i)^\perp \\
=& \{E \in \mathfrak{C}_i \mid \operatorname{Hom}(E', E) = 0, E' \in \widehat{\mathfrak{T}}_i\}.
\end{aligned}
\tag{4.22}
$$

Remark 4.3.3. We have $\widehat{\mathfrak{F}}_i \supset \overline{\mathfrak{F}}_i$ and $\widehat{\mathfrak{T}}_i \subset \overline{\mathfrak{T}}_i$.

Definition 4.3.4. $(\overline{\mathfrak{T}}_i, \overline{\mathfrak{F}}_i)$ and $(\widehat{\mathfrak{T}}_i, \widehat{\mathfrak{F}}_i)$ are torsion pairs of $\mathfrak{C}_i$. We denote the tiltings by $\overline{\mathfrak{A}}_i$ and $\widehat{\mathfrak{A}}_i$ respectively.

Then we have the following equivalence:

Proposition 4.3.5. *Ψ induces an equivalence $\overline{\mathfrak{A}}_1[-2] \to (\widehat{\mathfrak{A}}_2)_{op}$.*

For the proof of this proposition, we need the following properties.

Lemma 4.3.6. (1) *Assume that $E \in \overline{\mathfrak{T}}_1$. Then $\operatorname{Hom}(E, \mathcal{E}_{|\{x'\}\times X}) = 0$ for a general $x' \in X'$.*
(2) *Assume that $E \in \widehat{\mathfrak{F}}_1$. Then $\operatorname{Hom}(\mathcal{E}_{|\{x'\}\times X}, E) = \operatorname{Hom}(E_{ij}, E) = 0$ for all $x' \in X'$. In particular if $E \in \overline{\mathfrak{F}}_1$, then $\operatorname{Hom}(\mathcal{E}_{|\{x'\}\times X}, E) = \operatorname{Hom}(E_{ij}, E) = 0$ for all $x' \in X'$.*

Proof. We only prove (1). If $\operatorname{rk} E = 0$, then obviously the claim holds. Let E be a torsion free object on X such that $E_{|f}$ is a semi-stable locally free sheaf with $\chi(G_1, E_{|f}) = 0$ for a general f. Then if there is a non-zero homomorphism $\varphi : E \to \mathcal{E}_{|\{x'\}\times X}$, then φ is surjective and $E_{|f}$ is S-equivalent to $\mathcal{E}_{|\{x'\}\times X} \oplus \ker\varphi$, where $f = \mathfrak{p}^{-1}(\mathfrak{q}(x'))$. Therefore $\operatorname{Hom}(E, \mathcal{E}_{|\{x'\}\times X}) = 0$ for a general $x' \in \mathfrak{q}^{-1}(\mathfrak{p}(f)) \subset Y$. □

Lemma 4.3.7. *Let E be an object of $\mathcal{C} = \mathfrak{C}_1$.*
(1) *${}^pH^i(\Psi(E)) = 0$ for $i \geq 3$.*
(2) *$H^0({}^pH^2(\Psi(E))) = H^2(\Psi(E))$.*
(3) *${}^pH^0(\Psi(E)) \subset H^0(\Psi(E))$. In particular, ${}^pH^0(\Psi(E))$ is torsion free.*
(4) *If $\operatorname{Hom}(E, E_{ij}[2]) = 0$ for all i, j and $\operatorname{Hom}(E, \mathcal{E}_{|\{x'\}\times X}[2]) = 0$ for all $x' \in X'$, then ${}^pH^2(\Psi(E)) = 0$. In particular, if $E \in \widehat{\mathfrak{F}}_1$, then ${}^pH^2(\Psi(E)) = 0$.*
(5) *If E satisfies $E \in \overline{\mathfrak{T}}_1$, then ${}^pH^0(\Psi(E)) = 0$.*

Proof. By Lemma 4.2.2, $E \in \operatorname{Per}(X'/Y')$ is 0 if and only if $\operatorname{Hom}(E, A'_{ij}) = \operatorname{Hom}(E, \mathbb{C}_{x'}) = 0$ for all i, j and $x' \in X'$. Since

$$\begin{aligned} \operatorname{Hom}(E, E_{ij}[p]) &\cong \operatorname{Hom}(\Psi(E)[p], \Psi(E_{ij})(K_{X'})[2])^\vee \cong \operatorname{Hom}(\Psi(E)[p], \Psi(E_{ij})[2])^\vee, \\ \operatorname{Hom}(E, \mathcal{E}_{|\{x'\}\times X}[p]) &\cong \operatorname{Hom}(\Psi(E)[p], \Psi(\mathcal{E}_{|\{x'\}\times X})(K_{X'})[2])^\vee \cong \operatorname{Hom}(\Psi(E)[p], \Psi(\mathcal{E}_{|\{x'\}\times X})[2])^\vee, \end{aligned} \tag{4.23}$$

we have (1), (2) and (4). (3) is obvious. (5) follows from (3) and Lemma 4.3.6 (1). □

Corollary 4.3.8. *If $E \in \overline{\mathfrak{T}}_1 \cap \widehat{\mathfrak{F}}_1$, then ${}^pH^i(\Psi(E)) = 0$ for $i \neq 1$.*

Lemma 4.3.9. *Let E be an object of $\mathcal{C}$.*
(1) *If WIT_0 holds for E with respect to Ψ, then $E \in \overline{\mathfrak{F}}_1$.*
(2) *If WIT_2 holds for E with respect to Ψ, then $E \in \widehat{\mathfrak{T}}_1$.*

Proof. For an object E of $\mathcal{C}$, there is an exact sequence

$$0 \to E_1 \to E \to E_2 \to 0 \tag{4.24}$$

such that $E_1 \in \overline{\mathfrak{T}}_1$ and $E_2 \in \overline{\mathfrak{F}}_1$. Applying Ψ to this exact sequence, we get a long exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Psi^0(E_2) & \longrightarrow & \Psi^0(E) & \longrightarrow & \Psi^0(E_1) \\ & \longrightarrow & \Psi^1(E_2) & \longrightarrow & \Psi^1(E) & \longrightarrow & \Psi^1(E_1) \\ & \longrightarrow & \Psi^2(E_2) & \longrightarrow & \Psi^2(E) & \longrightarrow & \Psi^2(E_1) \longrightarrow 0. \end{array} \tag{4.25}$$

By Lemma 4.3.7, we have $\Psi^0(E_1) = \Psi^2(E_2) = 0$. If WIT_0 holds for E, then we get $\Psi(E_1) = 0$. Hence (1) holds. If WIT_2 holds for E, then we get $\Psi(E_2) = 0$. Thus $E \in \overline{\mathfrak{T}}_1$. We take a decomposition

$$0 \to E'_1 \to E \to E'_2 \to 0 \tag{4.26}$$

such that $E'_1 \in \widehat{\mathfrak{T}}_1$ and $E'_2 \in \widehat{\mathfrak{F}}_1 \cap \overline{\mathfrak{T}}_1$. Then $\Psi^i(E'_2) = 0$ for $i \neq 1$ by Corollary 4.3.8. Since $\Psi^0(E'_1) = 0$, we also get $\Psi^1(E'_2) = 0$. Therefore $E'_2 = 0$. □

Lemma 4.3.10. (1) *If $E \in \overline{\mathfrak{T}}_1$, then (1a) $\Psi^0(E) = 0$, (1b) $\Psi^1(E) \in \widehat{\mathfrak{F}}_2$ and (1c) $\Psi^2(E) \in \widehat{\mathfrak{T}}_2$.*
(2) *If $E \in \overline{\mathfrak{F}}_1$, then (2a) $\Psi^0(E) \in \widehat{\mathfrak{F}}_2$, (2b) $\Psi^1(E) \in \widehat{\mathfrak{T}}_2$ and (2c) $\Psi^2(E) = 0$.*

Proof. (1a) and (2c) follow from Lemma 4.3.7. (2a) is easy. (1c) By Lemma 6.3.2, WIT_2 holds for $\Psi^2(E)$ with respect to $\widehat{\Psi}$. By a similar claim of Lemma 4.3.9 (2), we get $\Psi^2(E) \in \widehat{\mathfrak{T}}_2$.

We next study $\Psi^1(E)$ for $E \in \mathcal{C}$. Assume that there is an exact sequence

$$0 \to F_1 \to \Psi^1(E) \to F_2 \to 0 \tag{4.27}$$

such that $F_1 \in \widehat{\mathfrak{T}}_2$ and $F_2 \in \widehat{\mathfrak{F}}_2$. Applying $\widehat{\Psi}$, we have a long exact sequence

$$
\begin{aligned}
0 &\longrightarrow \widehat{\Psi}^0(F_2) \longrightarrow \widehat{\Psi}^0(\Psi^1(E)) \longrightarrow \widehat{\Psi}^0(F_1) \\
&\longrightarrow \widehat{\Psi}^1(F_2) \longrightarrow \widehat{\Psi}^1(\Psi^1(E)) \longrightarrow \widehat{\Psi}^1(F_1) \\
&\longrightarrow \widehat{\Psi}^2(F_2) \longrightarrow \widehat{\Psi}^2(\Psi^1(E)) \longrightarrow \widehat{\Psi}^2(F_1) \longrightarrow 0.
\end{aligned} \tag{4.28}
$$

By Theorem 4.2.6, we have similar claims to Lemma 4.3.7. Thus we have $\widehat{\Psi}^0(F_1) = \widehat{\Psi}^2(F_2) = 0$.

Assume that $E \in \overline{\mathfrak{T}}_1$. Since $\Psi^0(E) = 0$, Lemma 6.3.2 implies that $\widehat{\Psi}^2(\Psi^1(E)) = 0$. Hence WIT_1 holds for F_1. Since $0 \le \chi(G_2, F_1) = \chi(\widehat{\Psi}^1(F_1), L_1) = -(H, c_1(\widehat{\Psi}^1(F_1))) \le 0$, $\widehat{\Psi}^1(F_1)$ is a 0-dimensional object. If $F_1 \ne 0$, then since $\widehat{\Psi}^1(F_1) \ne 0$, we see that $0 < \chi(G_1, \widehat{\Psi}^1(F_1)) = \chi(F_1, L_2) = -(\widehat{H}, c_1(F_1)) \le 0$, which is a contradiction. Therefore $F_1 = 0$.

Assume that $E \in \overline{\mathfrak{F}}_1$. Since $\Psi^2(E) = 0$, Lemma 6.3.2 implies that $\widehat{\Psi}^0(\Psi^1(E)) = 0$. Hence WIT_1 holds for F_2. We have an injection $\widehat{\Psi}^1(\Psi^1(E)) \to E$. Since $\mu_{\max, G_1}(E) < 0$, $\Psi^1(E)$ is zero on a generic fiber of $\mathfrak{p}$. Hence $\widehat{\Psi}^1(\Psi^1(E))$ is a torsion object. Since E is torsion free, $\widehat{\Psi}^1(\Psi^1(E)) = 0$. Since $\widehat{\Psi}^0(F_1) = 0$, we get $\widehat{\Psi}^1(F_2) = 0$, which implies that $F_2 = 0$. □

Proof of Proposition 4.3.5.

It is sufficient to prove that $\Psi(\overline{\mathfrak{T}}_1[-2]), \Psi(\overline{\mathfrak{F}}_1[-1]) \subset (\widehat{\mathfrak{A}}_2)_{op}$. Then the claims follow from Lemma 4.3.10. □

4.4. **Preservation of Gieseker stability conditions.** We give a generalization of [Y1, Thm. 3.15]. We first recall the following well-known fact.

Lemma 4.4.1. (1) *Let E be a torsion free object of $\mathcal{C}$. Then E is G_1-twisted semi-stable with respect to $H + nf$, $n \gg 0$ if and only if for every proper object E' of E, one of the following conditions holds:*

(a)

$$\frac{(c_1(E), f)}{\operatorname{rk} E} > \frac{(c_1(E'), f)}{\operatorname{rk} E'}, \tag{4.29}$$

(b)

$$\frac{(c_1(E), f)}{\operatorname{rk} E} = \frac{(c_1(E'), f)}{\operatorname{rk} E'}, \ \frac{(c_1(E), H)}{\operatorname{rk} E} > \frac{(c_1(E'), H)}{\operatorname{rk} E'}, \tag{4.30}$$

(c)

$$\frac{(c_1(E), f)}{\operatorname{rk} E} = \frac{(c_1(E'), f)}{\operatorname{rk} E'}, \ \frac{(c_1(E), H)}{\operatorname{rk} E} = \frac{(c_1(E'), H)}{\operatorname{rk} E'}, \ \frac{\chi(G_1, E)}{\operatorname{rk} E} \ge \frac{\chi(G_1, E')}{\operatorname{rk} E'}. \tag{4.31}$$

(2) *Let F be a 1-dimensional object of $\operatorname{Per}(X'/Y')$ with $(c_1(F), f) \ne 0$. Then F is G_2-twisted semi-stable with respect to $\widehat{H} + nf$, $n \gg 0$ if and only if for every proper subobject F' of F, one of the following conditions holds:*

(a)

$$(c_1(F'), f)\frac{\chi(G_2, F)}{(c_1(F), f)} > \chi(G_2, F') \tag{4.32}$$

(b)

$$(c_1(F'), f)\frac{\chi(G_2, F)}{(c_1(F), f)} = \chi(G_2, F'), \ (c_1(F'), \widehat{H})\frac{\chi(G_2, F)}{(c_1(F), \widehat{H})} > \chi(G_2, F'). \tag{4.33}$$

Lemma 4.4.2. *Let F be a purely 1-dimensional G_2-twisted semi-stable object such that $(c_1(F), f) > 0$ and $\chi(G_2, F) < 0$. Then WIT_1 holds for F with respect to $\widehat{\Psi}$ and $\widehat{\Psi}^1(F)$ is torsion free.*

Proof. By Lemma 4.4.1 (2), $F \in \widehat{\mathfrak{F}}_2$. By Theorem 4.2.6, similar claims to Lemma 4.3.7, Corollary 4.3.8 and Lemma 4.3.9 hold for $\widehat{\Psi}$. Hence WIT_1 holds for F. Assume that there is an exact sequence

$$0 \to E_1 \to \widehat{\Psi}^1(F) \to E_2 \to 0 \tag{4.34}$$

such that E_1 is the torsion object of $\widehat{\Psi}^1(F)$. Since $\widehat{\Psi}^1(F)_{|f}$ is a semi-stable vector bundle of $\deg(G_1^\vee \otimes \widehat{\Psi}^1(F)_{|f}) = 0$ for a general fiber f of $\mathfrak{p}$, $\operatorname{Supp}(E_1)$ is contained in fibers. Since $E_1 \in \overline{\mathfrak{T}}_1$ and $E_2 \in \widehat{\mathfrak{F}}_1$, WIT_1 holds for E_1, E_2 and we have a quotient $F \to \Psi^1(E_1)$. By our assumption on F, we get $\chi(G_2, \Psi^1(E_1)) \ge 0$. On the other hand, $\chi(G_2, \Psi^1(E_1)) = \chi(E_1, L_1) = -(H, c_1(E_1)) \le 0$. Hence E_1 is a 0-dimensional object. Then we get $0 < \chi(G_1, E_1) = \chi(\Psi^1(E_1), L_2) = -(\widehat{H}, c_1(\Psi^1(E_1))) \le 0$, which is a contradiction. □

Lemma 4.4.3. *Let F be a 1-dimensional object of* $\operatorname{Per}(X'/Y')$. *Then*

$$\begin{aligned}(c_1(F), f) &= \operatorname{rk}(\widehat{\Psi}(F)[1]),\\ (c_1(F), \widehat{H}) &= -\chi(F, L_2) = -\chi(G_1, \widehat{\Psi}(F)[1]),\\ \chi(G_2, F) &= \chi(\widehat{\Psi}(F)[1], L_1) = -(c_1(\widehat{\Psi}(F)[1]), H) + \operatorname{rk}(\widehat{\Psi}(F)[1])\chi(L_1).\end{aligned} \tag{4.35}$$

Proposition 4.4.4. *Let $w \in K(X')_{\text{top}}$ be a topological invariant of a 1-dimensional object. Assume that $\chi(G_2, w) < 0$. Then for $n \gg 0$, we have an isomorphism*

$$\mathcal{M}_{H+nf}^{G_1}(\widehat{\Psi}(-w))^{ss} \to \mathcal{M}_{\widehat{H}+nf}^{G_2}(w)^{ss}, \tag{4.36}$$

which preserves the S-equivalence classes.

Proof. Let E be a G_1-twisted semi-stable object with $\tau(E) = \widehat{\Psi}(-w)$. Then since $E_{|f}$ is a semi-stable locally free sheaf with $d \operatorname{rk} E - r \deg(E_{|f}) = 0$ for a general fiber, we have $E \in \overline{\mathfrak{T}}_1 \cap \widehat{\mathfrak{F}}_1$. By Corollary 4.3.8, WIT_1 holds for E with respect to Ψ. Assume that there is an exact sequence

$$0 \to F_1 \to \Psi^1(E) \to F_2 \to 0. \tag{4.37}$$

By Lemma 4.3.10, $\Psi^1(E) \in \widehat{\mathfrak{F}}_2$, which implies that $F_1 \in \widehat{\mathfrak{F}}_2$. Since $\operatorname{rk} \Psi^1(E) = 0$, $F_1, F_2 \in \overline{\mathfrak{T}}_2$. In particular, $F_1 \in \overline{\mathfrak{T}}_2 \cap \widehat{\mathfrak{F}}_2$. Then similar claim to Corollary 4.3.8 implies that WIT_1 holds for F_1. Hence we get an exact sequence

$$0 \to \widehat{\Psi}^1(F_2) \to E \overset{\varphi}{\to} \widehat{\Psi}^1(F_1) \to \widehat{\Psi}^2(F_2) \to 0. \tag{4.38}$$

By Lemma 4.3.10, $\widehat{\Psi}^2(F_2) \in \widehat{\mathfrak{T}}_1$. Hence $\operatorname{rk} \widehat{\Psi}^1(F_1) = \operatorname{rk} \operatorname{im} \varphi$. By (4.35), we have the following equivalences.

$$(c_1(F_1), f)\frac{\chi(G_2, \Psi^1(E))}{(c_1(F), f)} \leq \chi(G_2, F_1) \iff \operatorname{rk} \widehat{\Psi}^1(F_1)\frac{(c_1(E), H)}{\operatorname{rk} E} \geq (c_1(\widehat{\Psi}^1(F_1)), H), \tag{4.39}$$

$$(c_1(F_1), \widehat{H})\frac{\chi(G_2, \Psi^1(E))}{(c_1(\Psi^1(E)), \widehat{H})} \leq \chi(G_2, F_1) \iff -\chi(G_1, \widehat{\Psi}^1(F_1))\frac{\chi(G_2, \Psi^1(E))}{-\chi(G_1, E)} \leq \chi(G_2, F_1). \tag{4.40}$$

If the equality holds in (4.39), then $\chi(G_2, \Psi^1(E)) < 0$ implies that (4.40) is equivalent to

$$\frac{\chi(G_1, \widehat{\Psi}^1(F_1))}{\chi(G_1, E)} \geq \frac{\operatorname{rk} \widehat{\Psi}^1(F_1)}{\operatorname{rk} E} \tag{4.41}$$

which is equivalent to

$$\frac{\chi(G_1, \widehat{\Psi}^1(F_1))}{\operatorname{rk} \widehat{\Psi}^1(F_1)} \leq \frac{\chi(G_1, E)}{\operatorname{rk} E} \tag{4.42}$$

by $-\chi(G_1, E) > 0$. Since

$$\frac{\chi(G_1, \operatorname{im} \varphi(nH))}{\operatorname{rk} \operatorname{im} \varphi} \leq \frac{\chi(G_1, \widehat{\Psi}^1(F_1)(nH))}{\operatorname{rk} \widehat{\Psi}^1(F_1)},\ n \gg 0, \tag{4.43}$$

we see that φ is surjective and the equalities hold for (4.39), (4.40). Therefore $\Psi^1(E)$ is G_2-twisted semi-stable.

Conversely let F be a G_2-twisted semi-stable object with $\tau(F) = w$. By Lemma 4.4.2, WIT_1 holds for F with respect to $\widehat{\Psi}$ and $\widehat{\Psi}^1(F)$ is a torsion free object whose restriction to a general fiber is stable. If $\widehat{\Psi}^1(E)$ is not G_1-twisted semi-stable, then we have an exact sequence

$$0 \to E_1 \to \widehat{\Psi}^1(F) \to E_2 \to 0 \tag{4.44}$$

such that $E_i \in \overline{\mathfrak{T}}_1 \cap \widehat{\mathfrak{F}}_1$. By using Lemme 4.4.3, we get the following equivalences:

$$\frac{(c_1(\widehat{\Psi}^1(F)), H)}{\operatorname{rk} \widehat{\Psi}^1(F)} \leq \frac{(c_1(E_1), H)}{\operatorname{rk} E_1} \iff \frac{\chi(G_2, F)}{(c_1(F), f)} \geq \frac{\chi(G_2, \Psi^1(E_1))}{(c_1(\Psi^1(E_1)), f)}, \tag{4.45}$$

$$\frac{\chi(G_1, \widehat{\Psi}^1(F))}{\operatorname{rk} \widehat{\Psi}^1(F)} \leq \frac{\chi(G_1, E_1)}{\operatorname{rk} E_1} \iff \frac{(c_1(F), \widehat{H})}{(c_1(F), f)} \geq \frac{(c_1(\Psi^1(E_1)), \widehat{H})}{(c_1(\Psi^1(E_1)), f)}. \tag{4.46}$$

If the equality holds in (4.45), then (4.46) is equivalent to

$$\frac{\chi(G_2, F)}{(c_1(F), \widehat{H})} \geq \frac{\chi(G_2, \Psi^1(E_1))}{(c_1(\Psi^1(E_1)), \widehat{H})} \tag{4.47}$$

by $\chi(G_2, F) < 0$. Therefore $\widehat{\Psi}^1(F)$ is G_1-twisted semi-stable. □

5. A category of equivariant coherent sheaves.

5.1. **Morita equivalence for G-sheaves.** Let X be a smooth projective surface and G a finite group acting on X. Assume that $G \to \operatorname{Aut}(X)$ is injective and $\operatorname{Stab}(x)$, $x \in X$ acts trivally on $(K_X)_{|\{x\}}$, that is, K_X is the pull-back of a line bundle on $Y := X/G$. By our assumption, all elements of G have at most isolated fixed points sets. Let $R(G)$ be the representation ring of G and $(\ ,\ )$ the natural inner product. Let $K_G(X)$ be the Grothendieck group of G-sheaves and $K_G(X)_{\text{top}}$ its image to the Grothendieck group of topological G-vector bundles.

Definition 5.1.1. For G-sheaves E and F on X,

(1) $\operatorname{Ext}^i_G(E,F)$ is the G-invariant part of $\operatorname{Ext}^i(E,F)$.
(2) $\chi_G(E,F) := \sum_i (-1)^i \dim \operatorname{Ext}^i_G(E,F)$ is the Euler characteristic of the G-invariant cohomology groups of E, F. We also set $\chi_G(E) := \chi_G(\mathcal{O}_X, E)$.

Remark 5.1.2. If $K_X \cong \mathcal{O}_X$ in $\operatorname{Coh}_G(X)$, then $\chi_G(\ ,\ )$ is symmetric.

Let $\varpi : X \to Y$ be the quotient map. We set

$$\varpi_*(\mathcal{O}_X)[G] := \left\{ \sum_{g \in G} f_g(x) g \,\middle|\, f_g(x) \in \varpi_*(\mathcal{O}_X) \right\}. \tag{5.1}$$

$\varpi_*(\mathcal{O}_X)[G]$ is an $\mathcal{O}_Y$-algebra whose multiplication is defined by

$$(\sum_{g \in G} f_g(x) g) \cdot (\sum_{g' \in G} f'_{g'}(x) g') := \sum_{g, g' \in G} f_g(x) f'_{g'}(g^{-1}x) g g'. \tag{5.2}$$

We note that $\epsilon := \frac{1}{\#G} \sum_{g \in G} g$ satisfies $g\epsilon = \epsilon$ for all $g \in G$. By the injective homomorphism

$$\varpi_*(\mathcal{O}_X) \to \varpi_*(\mathcal{O}_X)\epsilon \ (\subset \varpi_*(\mathcal{O}_X)[G]), \tag{5.3}$$

we have an action of $\varpi_*(\mathcal{O}_X)[G]$ on $\varpi_*(\mathcal{O}_X)$:

$$(\sum_{g \in G} f_g(x) g) \cdot f(x) := \sum_{g \in G} f_g(x) f(g^{-1}x). \tag{5.4}$$

Thus we have a homomorphism

$$\varpi_*(\mathcal{O}_X)[G] \to \operatorname{Hom}_{\mathcal{O}_Y}(\varpi_*(\mathcal{O}_X), \varpi_*(\mathcal{O}_X)). \tag{5.5}$$

Lemma 5.1.3. $\varpi_*(\mathcal{O}_X)[G] \cong \operatorname{Hom}_{\mathcal{O}_Y}(\varpi_*(\mathcal{O}_X), \varpi_*(\mathcal{O}_X))$.

Proof. We first prove the claim over the smooth locus Y^{sm} of Y. We note that $\#\varpi^{-1}(y) = \#G$, $y \in Y^{\text{sm}}$. We take a point $z \in \varpi^{-1}(y)$. Then $\varpi_*(\mathcal{O}_X)_{|y} = \mathcal{O}_{\varpi^{-1}(y)}$ is identified with $\oplus_{g \in G} \mathbb{C}_{gz}$ as $\mathbb{C}[G]$-modules. Let $\chi_u(x)$ be the characteristic function of a point $u \in X$. Then $\{\chi_{gz} | g \in G\}$ is the base of $\oplus_{g \in G} \mathbb{C}_{gz}$ and $f(x) \in \mathcal{O}_{\varpi^{-1}(y)}$ is decomposed into $f(x) = \sum_{g \in G} f(gz)\chi_{gz}(x)$. Since

$$(\chi_{g'z}(x)(g'g^{-1})) \cdot (\sum_{h \in G} f(hz)\chi_{hz}(x)) = f(gz)\chi_{g'z}(x), \tag{5.6}$$

we see that

$$(\varpi_*(\mathcal{O}_X)[G])_{|y} \to \operatorname{Hom}(\varpi_*(\mathcal{O}_X)_{|y}, \varpi_*(\mathcal{O}_X)_{|y}) \tag{5.7}$$

is an isomorphism. Since $\varpi_*(\mathcal{O}_X)[G]$ and $\operatorname{Hom}_{\mathcal{O}_Y}(\varpi_*(\mathcal{O}_X), \varpi_*(\mathcal{O}_X))$ are reflexive sheaves on Y, we get the claim. □

We set $\mathcal{A} := \varpi_*(\mathcal{O}_X)[G] \cong \operatorname{Hom}_{\mathcal{O}_Y}(\varpi_*(\mathcal{O}_X), \varpi_*(\mathcal{O}_X))$.

Lemma 5.1.4. *We have an equivalence*

$$\begin{array}{cccc} \varpi_* : & \operatorname{Coh}_G(X) & \cong & \operatorname{Coh}_{\mathcal{A}}(Y) \\ & E & \mapsto & \varpi_*(E) \end{array} \tag{5.8}$$

whose inverse is $\varpi^{-1} : \operatorname{Coh}_{\mathcal{A}}(Y) \to \operatorname{Coh}_G(X)$. *In particular, we have an isomorphism*

$$\operatorname{Hom}_G(E_1, E_2) = \operatorname{Hom}_{\mathcal{A}}(\varpi_*(E_1), \varpi_*(E_2)),\ E_1, E_2 \in \operatorname{Coh}_G(X). \tag{5.9}$$

Proof. Since the problem is local, we may assume that Y is affine. Then X is also affine. For $F \in \operatorname{Coh}_{\mathcal{A}}(Y)$, $H^0(Y, F)$ is a $H^0(Y, \varpi_*(\mathcal{O}_X))[G]$-module. Hence $H^0(X, \varpi^{-1}(F)) = H^0(Y, F)$ is a $H^0(X, \mathcal{O}_X)[G]$-module, which implies that $\varpi^{-1}(F) \in \operatorname{Coh}_G(X)$. Then it is easy to see that ϖ^{-1} is the inverse of ϖ_*. □

By Lemma 5.1.4, we have an equivalence $\varpi_* : \mathbf{D}_G(X) \to \mathbf{D}_{\mathcal{A}}(Y)$. In particular,

$$\chi_G(E_1, E_2) = \sum_i (-1)^i \dim \operatorname{Hom}_{\mathcal{A}}(\varpi_*(E_1), \varpi_*(E_2)[i]),\ E_1, E_2 \in \operatorname{Coh}_G(X). \tag{5.10}$$

For a representation $\rho : G \to GL(V_\rho)$ of G, we define a G-linearization on $\mathcal{O}_X \otimes V_\rho$ in a usual way. Thus we define the action of G on $\varpi_*(\mathcal{O}_X \otimes V_\rho)$ as

$$g \cdot (f(x) \otimes v) := f(g^{-1}x) \otimes gv,\ g \in G, f(x) \in \varpi_*(\mathcal{O}_X), v \in V_\rho. \tag{5.11}$$

Then $\mathcal{O}_X \otimes \mathbb{C}[G]$ is a G-sheaf such that $\varpi_*(\mathcal{O}_X \otimes \mathbb{C}[G]) = \mathcal{A}$ and we have a decomposition

$$\mathcal{O}_X \otimes \mathbb{C}[G] = \bigoplus_i (\mathcal{O}_X \otimes V_{\rho_i})^{\oplus \dim \rho_i}, \tag{5.12}$$

where ρ_i are irreducible representations of G.

Definition 5.1.5. For a G-sheaf E and a representation $\rho : G \to GL(V_\rho)$, $E \otimes \rho$ denotes the G-sheaf $E \otimes_{\mathcal{O}_X} (\mathcal{O}_X \otimes V_\rho)$.

Since $\varpi_*(\mathcal{O}_X \otimes \rho_i)$ are direct summands of $\mathcal{A}$, we get the following lemma.

Lemma 5.1.6. (1) *$\mathcal{A}_i := \varpi_*(\mathcal{O}_X \otimes \rho_i)$ are local projective objects of* $\operatorname{Coh}_{\mathcal{A}}(Y)$.
(2) *$\bigoplus_i \varpi_*(\mathcal{O}_X \otimes \rho_i)^{\oplus r_i}$ is a local projective generator of $\operatorname{Coh}_{\mathcal{A}}(Y)$ if and only if $r_i > 0$ for all i.*

For a local projective generator $\mathcal{B}$ of $\operatorname{Coh}_{\mathcal{A}}(Y)$, we set $\mathcal{A}' := \mathcal{H}om_{\mathcal{A}}(\mathcal{B}, \mathcal{B})$. Then we have an equivalence

$$\begin{array}{ccc} \operatorname{Coh}_{\mathcal{A}}(Y) & \to & \operatorname{Coh}_{\mathcal{A}'}(Y) \\ E & \mapsto & \mathcal{H}om_{\mathcal{A}}(\mathcal{B}, E). \end{array} \tag{5.13}$$

5.2. **Stability for G-sheaves.** Let α be an element of $R(G) \otimes \mathbb{Q}$.

Definition 5.2.1. Let $\mathcal{O}_X(1)$ be the pull-back of an ample line bundle on Y. A coherent G-sheaf E is α-stable, if E is purely d-dimensional and

$$\frac{\chi_G(F(n) \otimes \alpha^\vee)}{a_d(F)} < \frac{\chi_G(E(n) \otimes \alpha^\vee)}{a_d(E)},\ n \gg 0 \tag{5.14}$$

for all proper subsheaf $F \neq 0$, where $a_d(*)$ is the coefficient of n^d of the Hilbert polynomial $\chi_G(*(n))$. We also define the α-semi-stability as usual.

Remark 5.2.2. Assume that $\alpha = \sum_i r_i \rho_i$, $r_i > 0$. We set $\mathcal{B} := \oplus_i \mathcal{A}_i^{\oplus r_i}$ and $\mathcal{A}' := \mathcal{H}om_{\mathcal{A}}(\mathcal{B}, \mathcal{B})$. Under the equivalence

$$\begin{array}{ccc} \operatorname{Coh}_G(X) & \to & \operatorname{Coh}_{\mathcal{A}'}(Y) \\ E & \mapsto & \mathcal{H}om_{\mathcal{A}}(\mathcal{B}, \varpi_*(E)), \end{array} \tag{5.15}$$

$$\chi_G(E(n) \otimes \alpha^\vee) = \chi(\mathcal{H}om_{\mathcal{A}}(\mathcal{B}, \varpi_*(E))(n)) \tag{5.16}$$

implies that α-twisted stability of E corresponds to the stability of $\mathcal{A}'$-module $\mathcal{H}om_{\mathcal{A}}(\mathcal{B}, \varpi_*(E))$.

For a coherent G-sheaf E of dimension 0, we also have a refined notion of stability, which also comes from the stability of 0-dimensional objects in $\operatorname{Coh}_{\mathcal{A}}(Y)$.

Definition 5.2.3. Let ρ_{reg} be the regular representation of G. A coherent G-sheaf E of dimension 0 is $(\rho_{\text{reg}}, \alpha)$-stable, if

$$\frac{\chi_G(F \otimes \alpha^\vee)}{\chi_G(F \otimes \rho_{\text{reg}}^\vee)} < \frac{\chi_G(E \otimes \alpha^\vee)}{\chi_G(E \otimes \rho_{\text{reg}}^\vee)} \tag{5.17}$$

for a proper subsheaf $F \neq 0$.

By [S, Thm. 4.7] and Proposition 1.6.1, we get the following theorem.

Theorem 5.2.4. (1) *Assume that $n\alpha$ contains every irreducible representation for a sufficiently large n. Then there is a coarse moduli space $\overline{M}_H^\alpha(v)$ of α-semi-stable G-sheaves E with $v(E) = v$. $\overline{M}_H^\alpha(v)$ is a projective scheme. We denote the open subscheme consisting of α-stable G-sheaves by $M_H^\alpha(v)$.*
(2) *Assume that v is a 0-dimensional vector. Then there is a coarse moduli space $\overline{M}_H^{\rho_{\text{reg}},\alpha}(v)$ of $(\rho_{\text{reg}}, \alpha)$-semi-stable G-sheaves E with $v(E) = v$. $\overline{M}_H^{\rho_{\text{reg}},\alpha}(v)$ is a projective scheme. We denote the open subscheme consisting of $(\rho_{\text{reg}}, \alpha)$-stable G-sheaves by $M_H^{\rho_{\text{reg}},\alpha}(v)$.*
(3) *If $K_X \cong \mathcal{O}_X$ in $\operatorname{Coh}_G(X)$, then $M_H^\alpha(v)$ and $M_H^{\rho_{\text{reg}},\alpha}(v)$ are smooth of dimension $-\chi_G(v, v) + 2$ with holomorphic symplectic structures.*

Remark 5.2.5. There is another construction due to Inaba [In].

For a smooth point y of Y, let v_0 be the topological invariant of $\mathcal{O}_{\varpi^{-1}(y)}$.

Lemma 5.2.6. *A 0-dimensional G-sheaf E is v_0-twisted stable if and only if E is an irreducible object of* $\mathrm{Coh}_G(X)$.

Proof. Let E be a G-sheaf of dimension 0. Then $\chi_G(E\otimes v_0^\vee)/\chi_G(E\otimes \rho_{reg}^\vee)=1$. Hence the claim holds. □

Definition 5.2.7. Let $G\text{-Hilb}_X^\rho$ be the G-Hilbert scheme parametrizing 0-dimensional subschemes Z of X such that $H^0(X,\mathcal{O}_Z)\cong V_\rho$.

Let $\rho_0,\rho_1,\dots,\rho_n$ be the irreducible representations of G. Assume that ρ_0 is trivial. We take an α such that $(\alpha,v_0)=0$ and $(\alpha,\rho_i)<0$ for $i>0$.

Lemma 5.2.8. $M_H^{\rho_{\mathrm{reg}},\alpha}(v_0)=G\text{-Hilb}_X^{\rho_{\mathrm{reg}}}$*. In particular,* $M_H^{\rho_{\mathrm{reg}},\alpha}(v_0)\neq\emptyset$.

Proof. Let E be a G-sheaf with $v(E)=v_0$. Since $\chi_G(\mathcal{O}_X\otimes\rho_0,E)=1$, we have a homomorphism $\phi:\mathcal{O}_X\otimes\rho_0\to E$. Then $H^0(\operatorname{im}\phi)$ contains a trivial representation, which implies that $\chi_G(\mathcal{O}_X\otimes\rho_0,\operatorname{im}\phi)\geq 1$. We note that E belongs to $M_H^{\rho_{\mathrm{reg}},\alpha}(v_0)$ if and only if E does not contain a proper subsheaf F with $\chi_G(\mathcal{O}_X\otimes\rho_0,F)\geq 1$. Hence if $E\in M_H^{\rho_{\mathrm{reg}},\alpha}(v_0)$, then $\operatorname{im}\phi=E$, which implies that $E\in G\text{-Hilb}_X^{\rho_{\mathrm{reg}}}$. Conversely, if $E\in G\text{-Hilb}_X^{\rho_{reg}}$, then for a subsheaf F with $\chi_G(\mathcal{O}_X\otimes\rho_0,F)\geq 1$, $\operatorname{Hom}_G(\mathcal{O}_X\otimes\rho_0,F)\to\operatorname{Hom}_G(\mathcal{O}_X\otimes\rho_0,E)$ is isomorphic. Hence ϕ factors through F. Since E is generated by the image of ϕ, $F=E$. Thus E is stable. □

We set $X':=M_H^{\rho_{\mathrm{reg}},\alpha}(v_0)$. Let Y' be the normalization of $\overline{M}_H^{\rho_{\mathrm{reg}},0}(v_0)$. Then we have a morphism $\pi:X'\to Y'$.

Proposition 5.2.9. (1) $Y'\to\overline{M}_H^{\rho_{\mathrm{reg}},0}(v_0)$ *is a bijective morphism.*

(2) *Let $\{p_1,p_2,\dots,p_l\}$ be the set of singular points of Y'. Then each p_i corresponds to S-equivalence classes of properly v_0-twisted semi-stable G-sheaves. Let $\oplus_{j=0}^{s_i}E_{ij}^{\oplus a_{ij}}$ be the S-equivalence class corresponding to p_i. Then the matrix $(\chi_G(E_{ij},E_{ij'}))_{j,j'\geq 0}$ is of affine type $\tilde{A},\tilde{D},\tilde{E}$.*

(3) *We can assume that $a_{i0}=1$ for all i. Then p_i is a rational double point of type A,D,E according as the type of the matrix $(\chi_G(E_{ij},E_{ij'}))_{j,j'\geq 1}$.*

(4) *We assume that $a_{i0}=1$ for all i. For $j\neq 0$,*

$$C_{ij}:=\{x'\in X'\mid \operatorname{Hom}_G(E_{ij},\mathcal{E}_{|\{x'\}\times X})\neq 0\} \tag{5.18}$$

is a smooth rational curve and $\pi^{-1}(p_i)=\sum_{j>0}a_{ij}C_{ij}$.

Proof. Since $H^0(X,\mathcal{O}_{\mathcal{Z}_{x'}})\cong\mathbb{C}[G]$, $x'\in X'$, we have

$$\sum_j a_{ij}\,\chi_G(\mathcal{O}_X\otimes\rho_0,E_{ij})=\chi_G(\mathcal{O}_X\otimes\rho_0,\oplus_j E_{ij}^{\oplus a_{ij}})=1. \tag{5.19}$$

Hence we may assume that $a_{i0}=1$ and $H^0(X,E_{ij})$ does not contain a trivial representation, if $j\neq 0$. In particular, $\chi_G(E_{ij}\otimes\alpha^\vee)<0$ for $j>0$. Then the proof is similar to the proof of Theorem 2.2.17 and Lemma 2.2.18. □

5.3. **Fourier-Mukai transforms for G-sheaves.** Let $\mathcal{E}:=\mathcal{O}_{\mathcal{Z}}$ be the universal family and we consider the Fourier-Mukai transform:

$$\begin{array}{rccc}\Phi: & \mathbf{D}_G(X) & \to & \mathbf{D}(X') \\ & E & \mapsto & \mathbf{R}\pi_{X'*}(\mathcal{E}\otimes\pi_X^*(E))^G.\end{array} \tag{5.20}$$

Then

$$\begin{array}{rccc}\widehat{\Phi}: & \mathbf{D}(X') & \to & \mathbf{D}_G(X) \\ & F & \mapsto & \mathbf{R}\pi_{X*}(\mathcal{E}^\vee[2]\otimes\pi_{X'}^*(F))\end{array} \tag{5.21}$$

is the quasi-inverse of Φ.

We note that $p_{X'*}(\mathcal{O}_{\mathcal{Z}})$ is a locally free sheaf on X' with a G-action. We have a decomposition of $p_{X'*}(\mathcal{O}_{\mathcal{Z}})$ as G-sheaves:

$$p_{X'*}(\mathcal{O}_{\mathcal{Z}})=\oplus_i\Phi(\mathcal{O}_X\otimes\rho_i)\otimes\rho_i^\vee. \tag{5.22}$$

For a G-sheaf E of dimension 0, $E^\vee=\mathcal{E}xt^2(E,\mathcal{O}_X)[-2]$. Hence E is an irreducible object if and only if $E^\vee[2]$ is an irreducible object.

Lemma 5.3.1. *We set $F_{ij}:=E_{ij}^\vee[2]\in\mathrm{Coh}_G(X)$.*

(1)

$$\Phi(F_{ij})=\begin{cases}\mathcal{O}_{C_{ij}}(-1)[1], & j>0,\\ \mathcal{O}_{Z_i}, & j=0,\end{cases} \tag{5.23}$$

where $Z_i:=\sum_j a_{ij}C_{ij}$ is the fundamental cycle of p_i.

(2) *$\Phi(\mathcal{O}_X\otimes\rho_i)$ is a locally free sheaf of rank $\dim\rho_i$ on X'. In particular, $\Phi(\mathcal{O}_X\otimes\rho_0)=\mathcal{O}_{X'}$.*

(3) $\Phi(\mathcal{O}_X \otimes \rho_i)$ *is a full sheaf* ([E]).

Proof. Let U be a G-invariant open subscheme of X. Then $D := \operatorname{Supp}(p_{Y*}(\mathcal{Z} \cap (Y \times (X \setminus U))))$ is a proper closed subset of Y and $\mathcal{Z}_y \subset U$ if and only if $y \in Y \setminus D$. If $K_U = \mathcal{O}_U$ as a G-sheaf, then we see that $K_{Y \setminus D}$ is trivial. Since X has an open covering of these properties, by the Grauert-Riemenschneider vanishing theorem, $\mathbf{R}\pi_*(\mathcal{O}_{X'}) = \mathcal{O}_{Y'}$. Outside of the fixed point loci of the G-action, $\widehat{\Phi}(\mathcal{O}_{X'})$ coincides with $\mathcal{O}_X \otimes \rho_0$. Hence $\widehat{\Phi}(\mathcal{O}_{X'}) = \mathcal{O}_X \otimes \rho_0$. Therefore $\Phi(\mathcal{O}_X \otimes \rho_0) = \mathcal{O}_{X'}$. (2) is a consequence of (5.22). Then the proof of (1) is similar to the Fourier-Mukai transform on a $K3$ surface. (3) We note that

$$\begin{aligned} \operatorname{Hom}(\Phi(\mathcal{O}_X \otimes \rho_i), \mathcal{O}_{C_{jk}}(-1)) &= \operatorname{Hom}(\Phi(\mathcal{O}_X \otimes \rho_i), \Phi(F_{jk})[-1]) \\ &= \operatorname{Hom}_G(\mathcal{O}_X \otimes \rho_i, F_{jk}[-1]) = 0, \\ \operatorname{Ext}^1(\Phi(\mathcal{O}_X \otimes \rho_i), \mathcal{O}_{Z_j}) &= \operatorname{Ext}^1(\Phi(\mathcal{O}_X \otimes \rho_i), \Phi(F_{j0})) \\ &= \operatorname{Ext}^1_G(\mathcal{O}_X \otimes \rho_i, F_{j0}) = 0. \end{aligned} \tag{5.24}$$

Hence $\Phi(\mathcal{O}_X \otimes \rho_i)$ is a full sheaf. □

We have

$$\Phi(\mathcal{O}_X \otimes \rho_i)_{|C_{jk}} \cong \mathcal{O}_{C_{jk}}^{\oplus(\dim \rho_i - k_{ijk})} \oplus \mathcal{O}_{C_{jk}}(1)^{\oplus k_{ijk}}, \tag{5.25}$$

where

$$\begin{aligned} k_{ijk} :=& (c_1(\Phi(\mathcal{O}_X \otimes \rho_i)), C_{jk}) \\ =& \dim \operatorname{Ext}^1(\Phi(\mathcal{O}_X \otimes \rho_i), \Phi(F_{jk})) \\ =& \dim \operatorname{Hom}_G(\mathcal{O}_X \otimes \rho_i, F_{jk}). \end{aligned} \tag{5.26}$$

Proposition 5.3.2. Φ *induces an equivalence*

$$\operatorname{Coh}_G(X) \to {}^{-1}\operatorname{Per}(X'/Y'). \tag{5.27}$$

Proof. It is sufficient to prove $\Phi(E) \in {}^{-1}\operatorname{Per}(X'/Y')$ for $E \in \operatorname{Coh}_G(X)$. We first prove that $H^i(\Phi(E)) = 0$ for $i \neq -1, 0$. Let E be a G-sheaf on X. Then there is an equivariant locally free resolution of E:

$$0 \to V_{-2} \to V_{-1} \to V_0 \to E \to 0. \tag{5.28}$$

Since $\Phi(V_i)$ are locally free sheaves on X' and

$$0 \to \Phi(V_{-2}) \to \Phi(V_{-1}) \to \Phi(V_0) \tag{5.29}$$

is exact on $X' \setminus \cup_i Z_i$, we get $H^i(\Phi(E)) = 0$ for $i \neq -1, 0$ and $\operatorname{Supp}(H^{-1}(\Phi(E))) \subset \cup_i Z_i$. Then we have

$$\begin{aligned} \operatorname{Hom}(H^0(\Phi(E)), \mathcal{O}_{C_{ij}}(-1)) &= \operatorname{Hom}(\Phi(E), \Phi(F_{ij})[-1]) \\ &= \operatorname{Hom}_G(E, F_{ij}[-1]) = 0, \ j > 0, \\ \operatorname{Hom}(\mathcal{O}_{Z_i}, H^{-1}(\Phi(E))) &= \operatorname{Hom}(\Phi(F_{i0}), \Phi(E)[-1]) \\ &= \operatorname{Hom}_G(F_{i0}, E[-1]) = 0. \end{aligned} \tag{5.30}$$

Hence $\Phi(E) \in {}^{-1}\operatorname{Per}(X'/Y')$. □

Remark 5.3.3. By the proof of Proposition 5.3.2, $H^{-1}(\Phi(E)) = 0$ if E does not contain a non-zero 0-dimensional sub G-sheaf.

Proposition 5.3.4. *For* $\alpha = \sum_i r_i \rho_i$, $r_i > 0$, *we set* $P := \bigoplus_i \Phi(\mathcal{O}_X \otimes \rho_i)^{\oplus r_i}$.

(1) P *is a local projective generator of* ${}^{-1}\operatorname{Per}(X'/Y')$.

(2) *A* G*-sheaf* E *is* α*-twisted stable if and only if* $\Phi(E)$ *is* P*-twisted stable.*

Proof. Since

$$\chi(P, \Phi(F_{jk})) = \sum_i r_i \chi_G(\mathcal{O}_X \otimes \rho_i, F_{jk}) = \sum_i r_i (\rho_i, H^0(X, F_{jk})) > 0 \tag{5.31}$$

for all j, k, (1) holds by Lemma 5.3.1 (3). (2) is obvious. □

Lemma 5.3.5. $\overline{M}_H^{v_0}(v_0) \cong Y' \cong X/G$. *In particular,* $\overline{M}_H^{v_0}(v_0)$ *is a normal surface with rational double points.*

Proof. We shall first show that $\overline{M}_H^{v_0}(v_0) \cong Y'$. By Proposition 5.3.4, $\overline{M}_H^{v_0}(v_0)$ is isomorphic to the moduli of 0-dimensional objects E of ${}^{-1}\operatorname{Per}(X'/Y')$ with $v(E) = v(\mathbb{C}_x)$. By Lemma 2.2.12, we have the claim.

Let $\Delta \subset X \times X$ be the diagonal. Then $\mathcal{G} := \oplus_{g \in G} \mathcal{O}_{(1 \times g)^*(\Delta)}$ is a G-equivariant coherent sheaf on $X \times X$ which is flat over X. Since $v(\mathcal{G}_{\{x\} \times X}) = v_0$, we have a morphism $\eta : X \to \overline{M}_H^{v_0}(v_0)$. We note that $\mathcal{G}_{|\{x\} \times X} \cong \mathcal{G}_{|\{g(x)\} \times X}$ for all $g \in G$ and $\mathcal{G}_{|\{x\} \times X} \cong \mathcal{G}_{|\{y\} \times X}$ if and only if $y \in Gx$. Hence η is G-invariant and we get an injective morphism $X/G \to \overline{M}_H^{v_0}(v_0)$. It is easy to see that $X/G \to \overline{M}_H^{v_0}(v_0)$ is an isomorphism. □

Corollary 5.3.6. *We set $P := \Phi(\mathcal{O}_X \otimes \mathbb{C}[G])$ and $\mathcal{A}' := \pi_*(P^\vee \otimes P)$. Under the isomorphism $Y' \cong Y$, we have an isomorphism $\pi_*(P) \cong \varpi_*(\mathcal{O}_X)$. Hence we have an isomorphism $\mathcal{A} \cong \mathcal{A}'$ as $\mathcal{O}_{Y'}$-algebras and we have the following commutative diagram.*

$$\begin{array}{ccc} \mathrm{Coh}_G(X) & \xrightarrow{\Phi} & \mathrm{Per}(X'/Y') \\ {\scriptstyle \varpi_*}\downarrow & & \downarrow{\scriptstyle \mathbf{R}\pi_*(P^\vee \otimes (\ ))} \\ \mathrm{Coh}_{\mathcal{A}}(Y) & = & \mathrm{Coh}_{\mathcal{A}'}(Y) \end{array} \tag{5.32}$$

Proof. We set $R := \mathcal{O}_X \otimes \mathbb{C}[G]$. Since $\Phi(\mathcal{O}_X \otimes \mathbb{C}[G]) \cong \oplus_i \Phi(\mathcal{O}_X \otimes \rho_i)^{\oplus \dim \rho_i} \cong p_{X'*}(\mathcal{O}_{\mathcal{Z}})$, $\pi_*(P) \cong \pi_*(p_{X'*}(\mathcal{O}_{\mathcal{Z}}))$ is a reflexive sheaf. Since $\pi_*(p_{X'*}(\mathcal{O}_{\mathcal{Z}})) = \varpi_*(\mathcal{O}_X)$ on the smooth locus, we get an isomorphism $\pi_*(P) \cong \varpi_*(\mathcal{O}_X)$. Since $\mathcal{A}'$ is a reflexive sheaf on Y', we have $\mathcal{A}' \cong \mathrm{End}_{\mathcal{O}_{Y'}}(\pi_*(P))$. Therefore $\mathcal{A}' \cong \mathrm{End}_{\mathcal{O}_{Y'}}(\pi_*(P)) \cong \mathrm{End}_{\mathcal{O}_Y}(\varpi_*(\mathcal{O}_X)) \cong \mathcal{A}$.

Since $\varpi_*(R) = \mathcal{A}$ and every G-sheaf E has a locally free resolution

$$\cdots \to R(-n_{-2})^{\oplus N_{-2}} \to R(-n_{-1})^{\oplus N_{-1}} \to R(-n_0)^{\oplus N_0} \to E \to 0, \tag{5.33}$$

we get the commutative diagram. □

Assume that X' is a $K3$ surface. For a primitive isotropic Mukai vector v_0 on X', we set $X'' := M_H^w(v_0)$, where $v_0 := (r, \xi, a)$ is a primitive isotropic Mukai vector with $0 < (\xi, C_{ij})$ and $(\xi, \sum_j a_{ij} C_{ij}) < r$ for all i, j and $w \in K(X') \otimes \mathbb{Q}$ is sufficiently close to v_0. Assume that there is a universal family $\mathcal{F}$ on $X' \times X''$. Then $\mathcal{E}' := \widehat{\Phi}(\mathcal{F})$ is a flat family of stable G-sheaves and defines an equivalence $\Phi' : \mathbf{D}^G(X) \to \mathbf{D}(X'')$ such that $\Phi' = \Phi^{\mathcal{E}'}_{X' \to X''} \circ \Phi$.

5.4. **Irreducible objects of** $\mathrm{Coh}_G(X)$**.** We shall study irreducible objects of $\mathrm{Coh}_G(X)$. Let E be a G-sheaf of dimension 0. We may assume that $\mathrm{Supp}(E) = Gx$. Let H be the stabilizer of x and E_x the submodule of E whose support is x. Then E_x is a H-sheaf. We have a decomposition $H^0(X, E) = \oplus_{y \in Gx} H^0(X, E_y)$. Since $gH^0(X, E_x) = H^0(X, E_{gx})$, we have an isomorphism

$$H^0(X, E) \cong \mathbb{C}[G] \otimes_{\mathbb{C}[H]} H^0(X, E_x) \tag{5.34}$$

as G-modules. Then we have an equality of invariant subspaces:

$$H^0(X, E)^G = H^0(X, E_x)^H. \tag{5.35}$$

We shall prove that there is a bijection between

(a) $\mathfrak{G} := \{E \in \mathrm{Coh}_G(X) | \mathrm{Supp}(E) = Gx,\ \mathrm{Stab}(x) = H\}$ and
(b) $\mathfrak{H} := \{F \in \mathrm{Coh}_H(X) | \mathrm{Supp}(F) = x\}$.

We define $r : \mathfrak{G} \to \mathfrak{H}$ by sending $E \in \mathfrak{G}$ to $E_x \in \mathfrak{H}$. For $F \in \mathfrak{H}$, we set $K := \ker(H^0(X, F) \otimes \mathcal{O}_X \to F)$. Then

$$s(F) := (\mathbb{C}[G] \otimes_{\mathbb{C}[H]} H^0(X, F)) \otimes \mathcal{O}_X / \sum_{g \in G} g(K) \tag{5.36}$$

is a G-sheaf such that $s(F)_x = F$. Hence we have a map $s : \mathfrak{H} \to \mathfrak{G}$ with $r \circ s = \mathrm{id}_{\mathfrak{H}}$. For $E \in \mathfrak{G}$, we also see that $s(E_x) \cong E$, and hence $s \circ r = \mathrm{id}_{\mathfrak{G}}$. Therefore our claim holds.

If $H^0(X, F)$ is the regular representation of H, i.e., $H^0(X, F) \cong \mathbb{C}[H]$, then $H^0(X, E)$ is the regular representation of G. Then we see that E is irreducible in $\mathrm{Coh}_G(X)$ if and only if E_x is irreducible in $\mathrm{Coh}_H(X)$. Since $\mathrm{Supp}(E_x)$ is one point, it means that $H^0(X, E_x)$ is an irreducible representation of H and $E_x \cong H^0(X, E_x) \otimes \mathbb{C}_x$.

Lemma 5.4.1. *Each singular point $\bigoplus_j E_{ij}^{\oplus a_{ij}} \in \overline{M}_H^{v_0}(v_0)$ corresponds to an orbit Gx_i with $\mathrm{Stab}(x_i) \neq \{e\}$ and $(E_{ij})_{x_i} = \rho_{ij} \otimes \mathbb{C}_{x_i}$, where ρ_{ij} are irreducible representations of $\mathrm{Stab}(x_i)$. Moreover*

$$\chi_G(E_{ij}, E_{ij'}) = \chi_{\mathrm{Stab}(x_i)}(\rho_{ij} \otimes \mathbb{C}_x, \rho_{ij'} \otimes \mathbb{C}_x). \tag{5.37}$$

Proof. If $\mathrm{Supp}(E_{ij}) \neq \mathrm{Supp}(E_{i'j'})$, then $\chi(E_{ij}, E_{i'j'}) = 0$. Hence $\mathrm{Supp}(E_{ij}) = \mathrm{Supp}(E_{ij'})$ for all j, j'. Hence there is a point x_i such that $Gx_i = \mathrm{Supp}(E_{ij})$ for all j. Then the first part of the claim follows.

For the second claim, we note that $\chi_{\mathrm{Stab}(x_i)}((\oplus_{g \in G/\mathrm{Stab}(x_i)} \rho_{ij} \otimes \mathbb{C}_{gx_i})/\rho_{ij} \otimes \mathbb{C}_{x_i}, \rho_{ij'} \otimes \mathbb{C}_{x_i}) = 0$. By using an equivariant locally free resolution of E_{ij} and (5.35), we see that

$$\begin{aligned} \chi_G(E_{ij}, E_{ij'}) &= \chi_{\mathrm{Stab}(x_i)}(E_{ij}, (E_{ij'})_{x_i}) \\ &= \chi_{\mathrm{Stab}(x_i)}((E_{ij})_{x_i}, (E_{ij'})_{x_i}). \end{aligned} \tag{5.38}$$

□

Example 5.4.2. Let X be an abelian surface. Then $G = \mathbb{Z}_2$ acts on X as the multiplication by (-1). Then the moduli of stable G-sheaves on X is isomorphic to the moduli space of stable objects of ${}^{-1}\mathrm{Per}(\mathrm{Km}(X)/Y)$, where $Y = X/G$ and $\mathrm{Km}(X) \to Y$ is the Kummer surface associated to X.

6.1. **Elementary facts on lattices.**

Lemma 6.1.1. *Assume that $L \cong \mathbb{Z}^n$ has an integral bilinear form $(\ ,\ )$. Let v be a primitive elenent of L such that $(v,v)=0$, $(v,w)=(w,v)$ for any w. We set $v^\perp := \{x \in L \mid (v,x)=0\}$. Assume that $(\ ,\ )_{|v^\perp}$ is symmetric and there is an element $u \in L \otimes \mathbb{Q}$ such that $(u,v)=0$ and $(v^\perp \cap u^\perp)/\mathbb{Z}v$ is negative definite.*

(1) *If $v=\sum_{i=0}^s a_i v_i$, $a_i \in \mathbb{Z}_{>0}$ such that $v_i \in v^\perp \cap u^\perp$, $i=0,1,...,s$, $(v_i^2)=-2$ and $(v_i,v_j) \geq 0$ for $i \neq j$. Then the matrix $(-(v_i,v_j)_{i,j})$ is of affine type $\widetilde{A},\widetilde{D},\widetilde{E}$.*

(2) *If v has two expressions*

$$v=\sum_{i=0}^s a_i v_i = \sum_{i=0}^t a_i' v_i',\ a_i, a_i' \in \mathbb{Z}_{>0} \tag{6.1}$$

such that $v_i, v_i' \in v^\perp \cap u^\perp$, $(v_i^2)=((v_i')^2)=-2$ and $(w_1,w_2) \geq 0$ for different $w_1,w_2 \in V_1 \cup V_2$, where $V_1 := \{v_0,v_1,\dots,v_s\}$ and $V_2 := \{v_0',v_1',\dots,v_t'\}$. Then $V_1=V_2$ or $\oplus_i \mathbb{Z}v_i \perp \oplus_i \mathbb{Z}v_i'$.

Proof. (1) We first note that $v_0,v_1,...,v_s$ are linearly independent over $\mathbb{Q}$. We shall show that the dual graph of $\{v_0,v_1,...,v_s\}$ is connected. If we have a decomposition $v=(\sum_{i\in I_1} a_i v_i)+(\sum_{i \in I_2} a_i v_i)$ such that $(v_i,v_j)=0$ for $i \in I_1$, $j \in I_2$, then $0=(v^2)=(\sum_{i\in I_1} a_i v_i)^2+(\sum_{i\in I_2} a_i v_i)^2$. Hence $\sum_{i \in I_1} a_i v_i, \sum_{i\in I_2} a_i v_i \in \mathbb{Z}v$, which implies that the graph is connected. Then the standard arguments show the claim.

(2) $I := \{i \mid v_i' \in V_1\}$ and $J := \{i \mid v_i' \notin V_1\}$. Then $v=(\sum_{i\in I} a_i' v_i')+(\sum_{i \in J} a_i' v_i')$. If $i \in J$, then $0=(v_i,v)=\sum_j a_j(v_i',v_j) \geq 0$. Hence $v_i' \in (\oplus_i \mathbb{Z}v_i)^\perp$. Then $0=(v^2)=((\sum_{i\in I} a_i'v_i')^2)+((\sum_{i \in J} a_i' v_i')^2)$. Hence $\sum_{i\in I} a_i' v_i', \sum_{i\in J} a_i' v_i' \in \mathbb{Z}v$, which implies that $I=\emptyset$ or $J=\emptyset$. If $J=\emptyset$, then $V_2 \subset V_1$, and we see that $V_1=V_2$. If $I=\emptyset$, then all v_i' belong to $\oplus_i \mathbb{Z}v_i$. Thus $\oplus_i \mathbb{Z}v_i \perp \oplus_i \mathbb{Z}v_i'$. □

Example 6.1.2. Let X be s smooth projective surface and H a divisor on X with $(H^2)>0$. We set $L := \operatorname{ch}(K(X))$ and $(x,y) := -\int_X x^\vee y \operatorname{td}_X$, $x,y \in L$. Then $\varrho_X = \operatorname{ch}(\mathbb{C}_x)$ is primitive in L. Since $\mathbb{C}_x \otimes K_X \cong \mathbb{C}_x$, $(\varrho_X,x)=(x,\varrho_X)$. Moreover $(\ ,\ )_{|\varrho_X^\perp}$ is symmetric. Since $(\varrho_X^\perp \cap \operatorname{ch}(\mathcal{O}_H)^\perp)/\mathbb{Z}\varrho_X \cong \{D \in \operatorname{NS}(X)_f \mid (H,D)=0\}$, it is negative definite, where $\operatorname{NS}(X)_f$ is the torsion free quotient of $\operatorname{NS}(X)$.

6.2. **Existence of twisted semi-stable sheaves.**

Let X be a smooth projective surface and H an ample divisor on X. Let $\mathbf{e} \in K(X)_{\text{top}}$ be a toplogical invariant of a coherent sheaf on X.

Definition 6.2.1. A polarization H on X is general with respect to $\mathbf{e}$, if for every μ-semi-stable sheaf E with $\tau(E)=\mathbf{e}$ and a subsheaf $F \neq 0$ of E,

$$\frac{(c_1(F),H)}{\operatorname{rk} F} = \frac{(c_1(E),H)}{\operatorname{rk} E} \text{ if and only if } \frac{c_1(F)}{\operatorname{rk} F} = \frac{c_1(E)}{\operatorname{rk} E}. \tag{6.2}$$

If H is general with respect to $\mathbf{e}$, then the G-twisted semi-stability does not depend on the choice of G. The following is [M-W, Lem. 3.6]. For convenience' sake, we give a proof.

Lemma 6.2.2. *Assume that H is not general with respect to $\mathbf{e}$ and let ϵ be a sufficiently small $\mathbb{Q}$-divisor such that $H+\epsilon$ is general with respect to $\mathbf{e}$. Then there is a locally free sheaf G such that $\mathcal{M}_H^G(\mathbf{e})^{ss} = \mathcal{M}_{H+\epsilon}(\mathbf{e})^{ss}$.*

Proof. We set

$$\mathcal{F}(\mathbf{e}) := \left\{ F \subset E \;\middle|\; \begin{matrix} E \in \mathcal{M}_H(\mathbf{e})^{\mu\text{-}ss},\ E/F \text{ is torsion free} \\ (c_1(F),H)/\operatorname{rk} F = (c_1(E),H)/\operatorname{rk} E \end{matrix} \right\}. \tag{6.3}$$

Since $\mathcal{F}(\mathbf{e})$ is a bounded set, we have

$$B := \max\left\{ \left| \frac{\chi(E)}{\operatorname{rk} E} - \frac{\chi(F)}{\operatorname{rk} F} \right| \;\middle|\; (F \subset E) \in \mathcal{F}(\mathbf{e}) \right\} < \infty. \tag{6.4}$$

Assume that $N\epsilon \in \operatorname{NS}(X)$. Let G be a locally free sheaf such that $c_1(G)/\operatorname{rk} G = -m\epsilon$. If $m \geq (\operatorname{rk} \mathbf{e})^2 NB$, then for $(F \subset E) \in \mathcal{F}(\mathbf{e})$,

$$\frac{\chi(G,E(nH))}{\operatorname{rk} E} - \frac{\chi(G,F(nH))}{\operatorname{rk} F} = m\left(\frac{c_1(E)}{\operatorname{rk} E} - \frac{c_1(F)}{\operatorname{rk} F}, \epsilon\right) + \frac{\chi(E)}{\operatorname{rk} E} - \frac{\chi(F)}{\operatorname{rk} F} \geq 0 \tag{6.5}$$

if and only if

(1)

$$\left(\frac{c_1(E)}{\operatorname{rk} E} - \frac{c_1(F)}{\operatorname{rk} F}, \epsilon\right) \geq 0 \tag{6.6}$$

or

(2)

$$\frac{c_1(E)}{\operatorname{rk} E} - \frac{c_1(F)}{\operatorname{rk} F} = 0, \ \frac{\chi(E)}{\operatorname{rk} E} - \frac{\chi(F)}{\operatorname{rk} F} \geq 0, \tag{6.7}$$

which is the semi-stability of E with respect to $H + \epsilon$. Therefore the claim holds. □

Lemma 6.2.3. *Let (X, H) be a polarized $K3$ surface and $v = r + \xi + a\varrho_X$, $\xi \in \operatorname{NS}(X)$ a primitive Mukai vector with $\langle v^2 \rangle \geq -2$. Then there is a G-twisted semi-stable sheaf E with $v(E) = v$ for any G.*

Proof. If H is general with respect to v, then there is a stable sheaf E with $v(E) = v$ by [Y1, Thm. 8.1] and [Y5]. By Lemma 6.2.2, there is a locally free sheaf G_1 such that $\mathcal{M}_H^{G_1}(v)^{ss} = \mathcal{M}_H^{G_1}(v)^s \neq \emptyset$. For a G with $\mathcal{M}_H^G(v)^{ss} = \mathcal{M}_H^G(v)^s$, we use [Y2, Prop. 4.1]. If $\mathcal{M}_H^G(v)^{ss} \neq \mathcal{M}_H^G(v)^s$, then we can find a G' such that $c_1(G')/\operatorname{rk} G'$ is sufficiently close to $c_1(G)/\operatorname{rk} G$, $\mathcal{M}_H^{G'}(v)^{ss} = \mathcal{M}_H^{G'}(v)^s \neq \emptyset$ and $\mathcal{M}_H^{G'}(v)^{ss} \subset \mathcal{M}_H^G(v)^{ss}$. Thus the claim also holds.

□

6.3. **Spectral sequences.** Since $\widehat{\Phi}[2]$ and $\widehat{\Psi}$ are the inverses of Φ and Ψ respectively, we get the followng.

Lemma 6.3.1. *We have spectral sequences*

$$E_2^{p,q} = \Phi^p(\widehat{\Phi}^q(E)) \Rightarrow E_\infty^{p+q} = \begin{cases} E, & p+q = 2, \\ 0, & p+q \neq 2, \end{cases} \quad E \in \operatorname{Per}(X'/Y'), \tag{6.8}$$

$$E_2^{p,q} = \widehat{\Phi}^p(\Phi^q(F)) \Rightarrow E_\infty^{p+q} = \begin{cases} F, & p+q = 2, \\ 0, & p+q \neq 2, \end{cases} \quad F \in \mathcal{C}. \tag{6.9}$$

In particular,

- (i) $\Phi^p(\widehat{\Phi}^q(E)) = 0$, $p = 0, 1$.
- (ii) $\Phi^p(\widehat{\Phi}^q(E)) = 0$, $p = 1, 2$.
- (iii) *There is an injective homomorphism* $\Phi^0(\widehat{\Phi}^1(E)) \to \Phi^2(\widehat{\Phi}^0(E))$.
- (iv) *There is a surjective homomorphism* $\Phi^0(\widehat{\Phi}^2(E)) \to \Phi^2(\widehat{\Phi}^1(E))$.

Lemma 6.3.2. *We have spectral sequences*

$$E_2^{p,q} = \Psi^p(\widehat{\Psi}^{-q}(E)) \Rightarrow E_\infty^{p+q} = \begin{cases} E, & p-q = 0, \\ 0, & p-q \neq 0, \end{cases} \quad E \in \operatorname{Per}(X'/Y')^D, \tag{6.10}$$

$$E_2^{p,q} = \widehat{\Psi}^p(\Psi^{-q}(F)) \Rightarrow E_\infty^{p+q} = \begin{cases} F, & p-q = 0, \\ 0, & p-q \neq 0, \end{cases} \quad F \in \mathcal{C}. \tag{6.11}$$

In particular,

- (i) $\Psi^p(\widehat{\Psi}^2(E)) = 0$, $p = 0, 1$.
- (ii) $\Psi^p(\widehat{\Psi}^0(E)) = 0$, $p = 1, 2$.
- (iii) *There is an injective homomorphism* $\Psi^0(\widehat{\Psi}^1(E)) \to \Psi^2(\widehat{\Psi}^2(E))$.
- (iv) *There is a surjective homomorphism* $\Psi^0(\widehat{\Psi}^0(E)) \to \Psi^2(\widehat{\Psi}^1(E))$.

For a convenience of the reader, we give a proof of Lemma 6.3.2.

Proof. By the exact triangles

$$\Psi^{\leq 1}(E)[-1] \to \Psi(E) \to \Psi^2(E)[-2] \to \Psi^{\leq 1}(E) \tag{6.12}$$

and

$$\Psi^0(E) \to \Psi^{\leq 1}(E)[-1] \to \Psi^1(E)[-1] \to \Psi^0(E)[1], \tag{6.13}$$

we have exact triangles

$$\widehat{\Psi}(\Psi^{\leq 1}(E))[1] \leftarrow \widehat{\Psi}(\Psi(E)) \leftarrow \widehat{\Psi}(\Psi^2(E))[2] \leftarrow \widehat{\Psi}(\Psi^{\leq 1}(E)) \tag{6.14}$$

and

$$\widehat{\Psi}(\Psi^0(E)) \leftarrow \widehat{\Psi}(\Psi^{\leq 1}(E))[1] \leftarrow \widehat{\Psi}(\Psi^1(E))[1] \leftarrow \widehat{\Psi}(\Psi^0(E))[-1]. \tag{6.15}$$

Since $\widehat{\Psi}(\Psi(E)) = E$, we have exact sequences

$$
\begin{aligned}
0 \leftarrow \widehat{\Psi}^1(\Psi^{\leq 1}(E)) \leftarrow E \leftarrow \widehat{\Psi}^2(\Psi^2(E)) \leftarrow \widehat{\Psi}^0(\Psi^{\leq 1}(E)) \leftarrow 0,\\
\widehat{\Psi}^2(\Psi^{\leq 1}(E)) = \widehat{\Psi}^1(\Psi^2(E)) = \widehat{\Psi}^0(\Psi^2(E)) = 0,\\
0 \leftarrow \widehat{\Psi}^2(\Psi^1(E)) \leftarrow \widehat{\Psi}^0(\Psi^0(E)) \leftarrow \widehat{\Psi}^1(\Psi^{\leq 1}(E)) \leftarrow \widehat{\Psi}^1(\Psi^1(E)) \leftarrow 0,\\
\widehat{\Psi}^0(\Psi^{\leq 1}(E)) \cong \widehat{\Psi}^0(\Psi^1(E)),\\
\widehat{\Psi}^1(\Psi^0(E)) = \widehat{\Psi}^2(\Psi^0(E)) = 0.
\end{aligned} \tag{6.16}
$$

These give the data of the spectral sequence. □

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